On critical kernels

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Abstract

We propose a method for collapsing simplicial complexes in a symmetric manner. For that purpose, we introduce the notions of a *simple cell*, of an *essential face*, and the one of a *core of a cell*. Then, we define the *critical kernel* of a complex. Our main result is that the critical kernel of a given complex X is a collapse of X. We extend this result by giving a necessary and sufficient condition which characterizes a certain class of subcomplexes of X which contain the critical kernel of X. In particular, any complex which belongs to this class is homotopy equivalent to X.

1 Introduction

The operation of collapse leads to a notion of homotopy equivalence in discrete spaces, which is the so-called simple homotopy equivalence [4]. This operation was originally settled in the context of (finite) simplicial complexes [9]. A simplicial complex Y is an elementary collapse of a simplicial complex X if $Y = X \setminus \{f, g\}$, where f, g are two distinct faces of X such that f is maximal for X (under inclusion), and f is the only face which contains g; the face g is said to be free for X. We observe that, in general, it is not possible to remove simultaneously all free faces (and their corresponding maximal faces) from an object: the result could be not homotopy equivalent to the original object. In fact the operation of collapse is basically not symmetric.

In this paper, we propose a method for collapsing simplicial objects in a symmetric manner. For that purpose, we introduce the notions of a *simple cell*, of an *essential face*, and the one of a *core of a cell*. Then, we define the *critical kernel* of a complex. Our main result is that the critical kernel of a given complex X is a collapse of X, *i.e.*, it may be obtained from X by a sequence of elementary collapses. We extend this result by showing that, if Y belongs to a certain class of subcomplexes of X which contain the critical kernel of X, then Y is homotopy equivalent to X. At last, we give a necessary and sufficient condition for this class of subcomplexes.

It is worth pointing out that homotopic retractions of discrete objects have received a lot of attention in the field of image analysis [6]. Many algorithms for extracting different kinds of "skeletons" of an object have been proposed (*e.g.*, curvilinear or surfacic skeletons in the 3D cubic grid). These algorithms are often based on ad-hoc conditions for removing points in a symmetric manner while preserving the topology of the original object. In fact, critical kernels constitute a framework for such algorithms [2,3] which may be seen as a generalization of the one presented in [1]. It should be noted that all the results presented hereafter may be directly transposed to N-dimensional cubical complexes.

2 Simple cells

A *(finite simplicial) complex* X is a finite family composed of finite nonempty sets such that, if f is an element of X, then every nonempty subset of f is an element of X. Each element of a complex is *a face* of this complex. The *dimension* of a face f is the number of its elements minus one. The *dimension of a complex* is the largest dimension of its faces. We denote by $\mathbb{K}$ the collection of all complexes.

Let f be a finite nonempty set. We set $\hat{f} = \{g \mid g \subseteq f,\ g \neq \emptyset\}$ and $\hat{f}^* = \hat{f} \setminus \{f\}$. Any $g \in \hat{f}$ is a *face of* f, and any $g \in \hat{f}^*$ is a *proper face of* f. If X is a finite family composed of finite nonempty sets, we write $X^- = \cup\{\hat{f} \mid f \in X\}$, X^- is the *(simplicial) closure of* X. Thus, a finite family X of finite nonempty sets is a complex if and only if $X = X^-$.

A complex X is a *cell* if there exists a face $f \in X$, such that $X = \hat{f}$.

A family Y is a *subcomplex* of a complex X, written $Y \preceq X$, if Y is a complex and if $Y \subseteq X$.

Let $X \in \mathbb{K}$. A face $f \in X$ is a *facet of* X if there is no $g \in X$ such that $f \in \hat{g}^*$. We denote by X^+ the set composed of all facets of X. Thus, we have $[X^+]^- = X$.

Let $X, Y \in \mathbb{K}$. We set $X \oslash Y = [X^+ \setminus Y^+]^-$. The set $X \oslash Y$ is a complex which is the *detachment of* Y *from* X.

Intuitively a cell $\hat{f}$ in a complex X is simple if its detachment from X "does not change the topology of X". In this section we propose a definition of a simple cell based on the operation of collapse [5]. It may be seen as a discrete analogue of the one given by T.Y. Kong in [8] which lies on continuous deformations in the N-dimensional Euclidean space.

Let $X \in \mathbb{K}$ and let $f \in X^+$. The facet f is a *border face of* X if there exists one face $g \in \hat{f}^*$ such that f is the only face of X which contains g. Such a face g is a *free face of* X and the pair (f, g) is said to be a *free pair for* X. If (f, g) is a free pair for X, the complex $X \setminus \{f, g\}$ is an *elementary collapse of* X.

Let X, Y be two complexes. We say that X *collapses onto* Y if there exists a *collapse sequence from* X *to* Y, *i.e.*, a sequence of complexes $\langle X_0, ..., X_l \rangle$ such that $X_0 = X$, $X_l = Y$, and X_i is an elementary collapse of X_{i-1}, $i = 1, ..., l$.

Definition 2.1 *Let $X \in \mathbb{K}$ and let $\hat{f}$ be a cell. We say that $\hat{f}$ and f are* simple for X *if $X \cup \hat{f}$ collapses onto $X \oslash \hat{f}$.*

Let $X \in \mathbb{K}$. If $f \in X^+$, f is simple for X iff X collapses onto $X \oslash \hat{f}$; if $f \in X \setminus X^+$, then f is trivially simple for X. If $f \notin X$, f is simple for X iff $X \cup \hat{f}$ collapses onto X. If $f \in X^+$, we have $[X \oslash \hat{f}] \cup \hat{f} = X$, thus f is simple for X iff f is simple for $X \oslash \hat{f}$. If $f \notin X$, we may have $[X \cup \hat{f}] \oslash \hat{f} \neq X$, thus, in general, it is not true that f is simple for X iff f is simple for $X \cup \hat{f}$.

The notion of attachment, as introduced by T.Y. Kong [7,8], leads to a local characterization of simple cells.
Let $X \in \mathbb{K}$ and let $\hat{f}$ be a cell. The *attachment of $\hat{f}$ for X* is the complex $Attach(\hat{f}, X) = \hat{f} \cap [X \oslash \hat{f}]$. If $f \in X^+$, a face g is in $Attach(\hat{f}, X)$ iff g is in $\hat{f}^*$ and g is a face of a facet h of X distinct from f. If $f \notin X$, we have $Attach(\hat{f}, X) = \hat{f} \cap X$.

The following proposition is an easy consequence of the above definitions.

Proposition 2.2 *Let $X \in \mathbb{K}$ and let $\hat{f}$ be a cell. The cell $\hat{f}$ is simple for X if and only if $\hat{f}$ collapses onto $Attach(\hat{f}, X)$.*

3 Essential faces

We introduce the notion of an essential face and the one of a core on which are based critical kernels.

Definition 3.1 *Let $X \in \mathbb{K}$ and let $f \in X$. We say that f is* essential for X *if f is precisely the intersection of all facets of X which contain f, i.e., if $f = \cap\{g \in X^+ \mid f \subseteq g\}$. We denote by $Ess(X)$ the set composed of all faces which are essential for X.*
Let $X \in \mathbb{K}$. If $Y \preceq X$ and if $Ess(Y) \subseteq Ess(X)$, we write $Y \trianglelefteq X$.

Observe that a facet of X is necessarily essential for X, *i.e.*, $X^+ \subseteq Ess(X)$. Note also that any face which is the intersection of essential faces is itself essential. The following property may be easily checked.

Proposition 3.2 *Let $X \in \mathbb{K}$ and $Y \preceq X$. We have $Y \trianglelefteq X$ if and only if $Y^+ \subseteq Ess(X)$.*

Proposition 3.3 *Let $X, Y, Z \in \mathbb{K}$, with $X \trianglelefteq Z$. Then $X \oslash Y \trianglelefteq Z$. Furthermore, if $Y \trianglelefteq Z$, then we have $X \cup Y \trianglelefteq Z$, and $X \cap Y \trianglelefteq Z$.*

Proof:
We have $[X \oslash Y]^+ \subseteq X^+$, thus, if $X \trianglelefteq Z$, Prop. 3.2 gives $X \oslash Y \trianglelefteq Z$. We also have $[X \cup Y]^+ \subseteq X^+ \cup Y^+$, thus, if $X \trianglelefteq Z$ and $Y \trianglelefteq Z$, Prop. 3.2 gives $X \cup Y \trianglelefteq Z$. Now observe that, in general, we have not $[X \cap Y]^+ \subseteq X^+ \cap Y^+$. Let $f \in [X \cap Y]^+$. We set $f_1 = \cap\{g \in X^+ \mid f \subseteq g\}$, $f_2 = \cap\{g \in Y^+ \mid f \subseteq g\}$, and $f' = f_1 \cap f_2$. We have $f \subseteq f'$. We note that f' is a face of $X \cap Y$. Thus, since f is a facet of $X \cap Y$, we have $f = f'$. If $X \trianglelefteq Z$ and if $Y \trianglelefteq Z$, f_1 and f_2 are both essential for Z, so is f. In this case, by Prop. 3.2, we have $X \cap Y \trianglelefteq Z$. $\square$

Definition 3.4 *Let $X \in \mathbb{K}$ and let $f \in Ess(X)$. The* core of $\hat{f}$ for *X is the complex, denoted by $Core(\hat{f}, X)$, such that $Core(\hat{f}, X) = \cup\{\hat{g} \mid g \in Ess(X) \cap \hat{f}^*\}$.*

Proposition 3.5 *Let $X \in \mathbb{K}$ and let $f \in X^+$. The attachment of $\hat{f}$ for X is precisely the core of $\hat{f}$ for X, i.e, we have $Attach(\hat{f}, X) = Core(\hat{f}, X)$.*

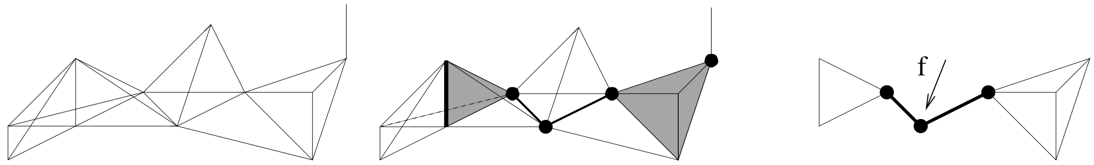
Proof:
- Let $g \in Attach(\hat{f}, X)$. By the very definition of the attachment, there exists a facet $h \neq f$ such that $g \subseteq h$. Thus, $h \cap f$ is not empty, $h \cap f$ is a face which is essential for X and which is in $\hat{f}^*$, hence $h \cap f \in Core(\hat{f}, X)$. Since $g \subseteq h \cap f$, we have $g \in Core(\hat{f}, X)$.
- Let $g \in Core(\hat{f}, X)$. There exists an essential face $h \in Core(\hat{f}, X)$, with $g \subseteq h$. By the very definition of an essential face, and since h is not a facet, it means that there exists at least one facet distinct from f which contains h and g. Thus $g \in Attach(\hat{f}, X)$. $\square$

Proposition 3.6 *Let $X \in \mathbb{K}$ and let $f \in Ess(X)$. Let $Y = \cup\{\hat{h} \mid h \in X^+$ and $f \subseteq h\}$ and let $g \in \hat{f}^*$. The face g is essential for X if and only if g is essential for $Z = (X \oslash Y) \cup \hat{f}$. Thus $Core(\hat{f}, X) = Core(\hat{f}, Z)$.*

Proof:
- Suppose $g \in Ess(X)$. Let $g_1 = \cap\{h \in X^+ \mid f \subseteq h\}$ and $g_2 = \cap\{h \in X^+ \mid g \subseteq h$ and $f \not\subseteq h\}$. We have $g = \cap\{h \in X^+ \mid g \subseteq h\} = g_1 \cap g_2$. But, since f is essential for X, $g_1 = f$, thus $g = f \cap g_2$. Since $f \in Z^+$ and since any $h \in X^+$ such that $f \not\subseteq h$ is in Z^+, this shows that g is essential for Z.
- We have $Z^+ = [X^+ \setminus Y^+] \cup \{f\}$. This implies $Z \trianglelefteq X$: if g is essential for Z, then g is essential for X. $\square$

4 Critical kernels

We are now in position to define the critical kernel of a complex, see illustration Fig. 1.



(a) (b) (c)

Figure 1. (a): A complex X_0. (b): The critical faces of X_0 are highlighted. (c) The complex $X_1 = Critic(X_0)$, the critical faces of X_1 are highlighted. The complex $X_2 = Critic(X_1)$ is such that $X_2 = \{f\}$. At last, we have $Critic(X_2) = X_2$.

Definition 4.1 *Let $X \in \mathbb{K}$. A face $f \in Ess(X)$ is* regular for X *if $\hat{f}$ collapses onto $Core(\hat{f}, X)$. A face $f \in Ess(X)$ is* critical for X *if f is not regular for X. We set $Critic(X) = \cup\{\hat{f} \mid f$ is critical for $X\}$, $Critic(X)$ is the* critical kernel of X.

By Prop. 3.5 and 2.2, a facet f of X is regular for X if and only if it is simple for X. The following property extends this fact, it is a direct consequence of Prop. 3.6.

Proposition 4.2 *Let $X \in \mathbb{K}$ and $f \in Ess(X)$. Let $Y = \cup\{\hat{g} \mid g \in X^+$ and $f \subseteq g\}$ and $Z = (X \oslash Y) \cup \hat{f}$. The face f is regular for X if and only if it is simple for Z.*

Theorem 4.3 *Let $X \in \mathbb{K}$.*
i) The complex X collapses onto its critical kernel.
ii) If $Y \trianglelefteq X$ contains the critical kernel of X, then X collapses onto Y.
iii) If $Y \trianglelefteq X$ contains the critical kernel of X, then any Z such that $Y \preceq Z \trianglelefteq X$ collapses onto Y.

Proof: We prove property iii), iii) implies ii) and, since $Critic(X) \trianglelefteq X$, ii) implies i).
Suppose that $Y \trianglelefteq X$ and that Y contains the critical kernel of X. Let Z such that $Y \preceq Z \trianglelefteq X$. Let f be a face in $Z^+ \setminus Y$, and let $Z' = [Z \oslash \hat{f}] \cup Core(\hat{f}, X)$. We have $Z' \preceq Z$, and $Z' \neq Z$ (since

$f \notin Z'$). Note that if no such a face f exists, then $Z = Y$. Observe also that we have not necessarily $Z' \trianglelefteq Z$.

1) By Prop. 3.3, since $Z \trianglelefteq X$, we have $Z \oslash \hat{f} \trianglelefteq X$. Again by Prop. 3.3, since $Core(\hat{f}, X) \trianglelefteq X$, we have $Z' \trianglelefteq X$.

2) We have $Core(\hat{f}, X) \subseteq \hat{f} \cap Z'$. Since $\hat{f} \trianglelefteq X$ and $Z' \trianglelefteq X$, by Prop. 3.3, $\hat{f} \cap Z' \trianglelefteq X$. Consequently $[\hat{f} \cap Z']^+ \subseteq Ess(X)$ and, since $\hat{f} \cap Z' \subseteq \hat{f}^*$, it implies that $\hat{f} \cap Z' \subseteq Core(\hat{f}, X)$. Thus, $\hat{f} \cap Z' = Core(\hat{f}, X)$. Since $Critic(X) \subseteq Y$, and $f \notin Y$, the face f is regular for X, it means that $\hat{f}$ collapses onto $Core(\hat{f}, X)$. Thus, since $Attach(\hat{f}, Z') = \hat{f} \cap Z' = Core(\hat{f}, X)$, $\hat{f}$ collapses onto $Attach(\hat{f}, Z')$. By Prop. 2.2, this implies that $\hat{f}$ is simple for Z', *i.e.*, $Z' \cup \hat{f}$ collapses onto Z'. But $Z = Z' \cup \hat{f}$ and so Z collapses onto Z'.

3) Let $g \in Y^+$. Accordingly, we have $g \in Z$. As $Z = Z' \cup \hat{f}$, either $g \in Z'$, or $g \in \hat{f}$. Suppose $g \in \hat{f}$. Clearly $g \neq f$ (since $f \in Z^+ \setminus Y$) and $g \in \hat{f}^*$. But, g being essential for X, we see that $g \in Core(\hat{f}, X)$. The result is $g \in Z'$. Thus, $Y^+ \subseteq Z'$, it follows that $Y \preceq Z'$.

By iteratively performing the operation $Z \rightarrow Z'$, the property is proved by induction. □

Definition 4.4 *Let* $X \in \mathbb{K}$ *and* $Y \trianglelefteq X$. *We say that* Y *is a* strong collapse of X *if* Z *collapses onto* T *whenever* $Y \preceq T \preceq Z \preceq X$, $T \trianglelefteq X$, *and* $Z \trianglelefteq X$.

Theorem 4.5 *Let* $X \in \mathbb{K}$, *and* $Y \trianglelefteq X$. *The complex* Y *is a strong collapse of* X *if and only if* Y *contains the critical kernel of* X.

Proof:
i) If Y contains the critical kernel of X, then, by Th. 4.3 iii), Y is a strong collapse of X.
ii) Let Y be a strong collapse of X. Suppose Y does not contain the critical kernel of X. It means that there exists $f \in X \setminus Y$ which is critical for X. Let $Z = Y \cup \hat{f}$ and let $T = Y \cup Core(\hat{f}, X)$.

By Prop. 3.3, $Z \trianglelefteq X$ and $T \trianglelefteq X$. We have $Z = T \cup \hat{f}$ and $Attach(\hat{f}, T) = \hat{f} \cap T$. Clearly $Core(\hat{f}, X) \subseteq \hat{f} \cap T$. Since $\hat{f} \trianglelefteq X$ and $T \trianglelefteq X$, by Prop. 3.3, $\hat{f} \cap T \trianglelefteq X$. Consequently $[\hat{f} \cap T]^+ \subseteq Ess(X)$ and, since $\hat{f} \cap T \subseteq \hat{f}^*$, we must have $\hat{f} \cap T \subseteq Core(\hat{f}, X)$. Thus, $\hat{f} \cap T = Core(\hat{f}, X) = Attach(\hat{f}, T)$. Since f is critical for X, it implies that f does not collapse onto $Attach(\hat{f}, T)$: f would not be simple for T (Prop. 2.2), and, by the very definition of a simple face, $T \cup \hat{f} = Z$ would not collapse onto T, a contradiction. □

References

[1] G. Bertrand, "On P-simple points", *Comptes Rendus de l'Académie des Sciences, Série Math.*, I, **321**, 1077-1084, 1995.

[2] G. Bertrand and M. Couprie, "Two-dimensional parallel thinning algorithms based on critical kernels", *Internal Report, Université de Marne-la-Vallée*, IGM2006-02, Also submitted for publication, 2006.

[3] G. Bertrand and M. Couprie, "Three-dimensional parallel thinning algorithms based on critical kernels", *Lecture Notes in Computer Science*, **4245**, 580-591, 2006.

[4] M. M. Cohen, *A course in simple-homotopy theory*, Springer Verlag, 1973.

[5] P. Giblin, *Graphs, surfaces and homology*, Chapman and Hall, 1981.

[6] R. Klette and A. Rosenfeld, *Digital geometry, geometric methods for digital picture analysis*, Morgan Kaufmann, 2006.

[7] T. Y. Kong, "On topology preservation in 2-D and 3-D thinning", *International Journal on Pattern Recognition and Artificial Intelligence*, **9**, 813-844, 1995.

[8] T. Y. Kong, "Topology-preserving deletion of 1's from 2-, 3- and 4-dimensional binary images", *Lecture Notes in Computer Science*, **1347**, 3-18, 1997.

[9] J.H.C. Whitehead, "Simplicial spaces, nuclei, and m-groups", *Proc. London Math. Soc.*, (2) **45**, 243-327, 1939.