A New 3D Parallel Thinning Scheme Based on Critical Kernels

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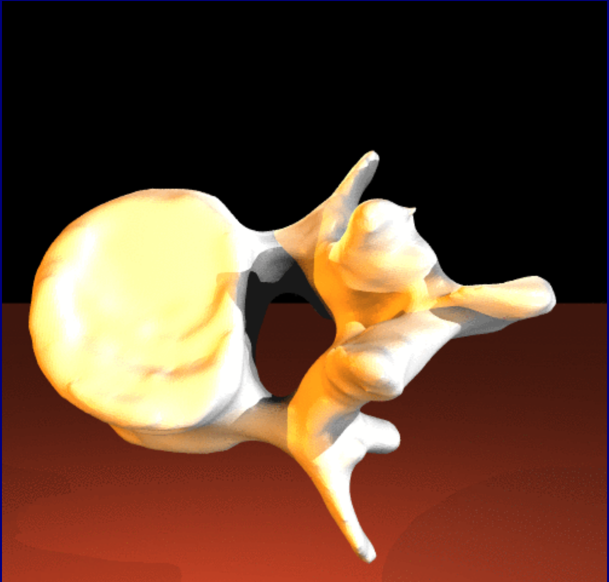
Parallel 3D thinning



Milestones

- 1966: D. Rutovitz – first parallel thinning algorithm

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Milestones

- Since then, many parallel thinning algorithms have been proposed, described by sets of masks

Milestones

- 1988: C. Ronse – minimal non-simple sets
- allows checking the topological soundness of parallel thinning algorithms

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Milestones

2005: Critical kernels

Plan of the presentation

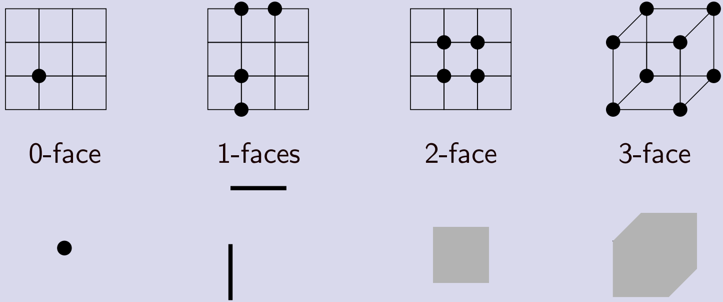
- Cubical complexes and simple faces
- Critical kernels
- New 3D Parallel Thinning Scheme
- Local characterizations and thinning algorithms
- Conclusion and perspectives

Part I

Cubical complexes

Face

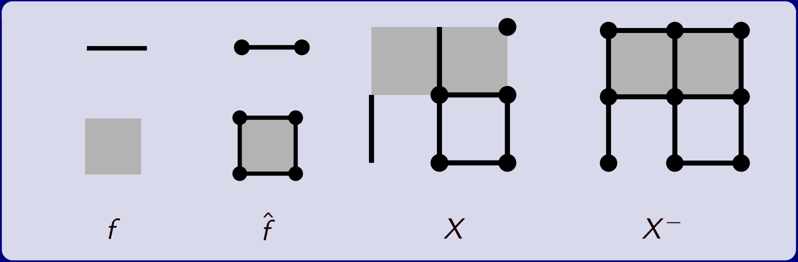
- A subset of $\mathbb{Z}^n$ composed of one point is called a 0-face.
- A subset of $\mathbb{Z}^n$ forming a unit bipoint is called a 1-face.
- A subset of $\mathbb{Z}^n$ forming a unit square is called a 2-face.
- A subset of $\mathbb{Z}^n$ forming a unit cube is called a 3-face.



Closure

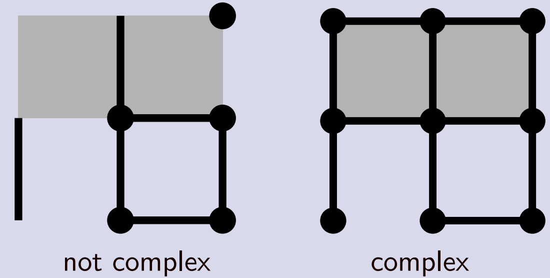
Let f be a face.

- The closure of f, denoted by $\hat{f}$, is the set composed by all the faces which are included in f.
- The set $\hat{f}$ is called a cell.
- If X is a finite set of faces, we write $X^- = \cup\{\hat{f} \mid f \in X\}$, X^- is the closure of X.



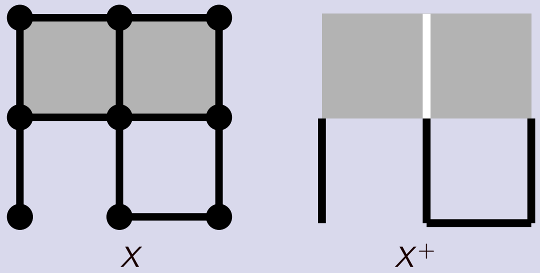
Cubical complex

A finite set X of faces is a complex if $X = X^-$.



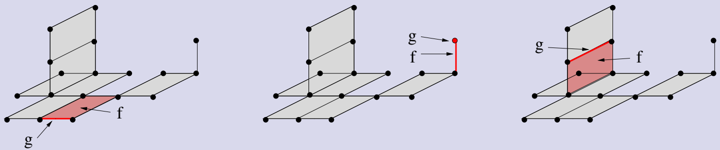
Principal face

- A face $f \in X$ is principal if there is no $g \in X$ such that f is strictly included in g.
- We denote by X^+ the set composed of all principal faces of X.



Elementary collapse

- Let f and g be two distinct faces such that f is the only face of X which contains g.
- The complex $X \setminus \{f, g\}$ is an elementary collapse of X.



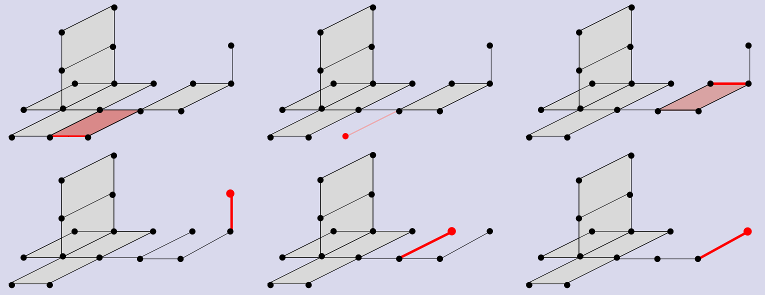
(a) (b) (c)

In (a) and (b), $X \setminus \{f, g\}$ is an elementary collapse of X.
In (c), $X \setminus \{f, g\}$ is not an elementary collapse of X.

Important: collapse preserves topology

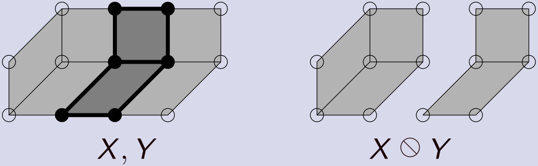
Collapse sequence, retraction

- Let X, Y be two complexes. We say that X collapses onto Y if there exists a collapse sequence from X to Y.
- If X collapses onto Y, we also say that Y is a retraction of X.



Detachment

Let Y be a subset of X. We set $X \oslash Y = [X^+ \setminus Y^+]^-$. The set $X \oslash Y$ is a complex which is the detachment of Y from X.

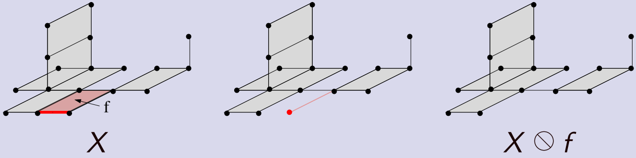


New definition for simplicity

Intuitively, a face f of a complex X is simple if its removal from X "does not change the topology of X".

Definition

Let f be a principal face, we say that f is simple if X collapses onto $X \oslash f$.



The face f is simple.

Part II

Critical kernels

Key notions

This framework for the study of parallel thinning in any dimension is based on three new notions:

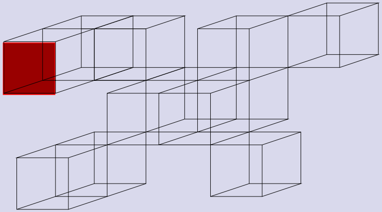
- Essential face
- Core of a face
- Regular/critical face

Essential face

Definition

- We say that f is an essential face if f is precisely the intersection of all principal faces of X which contain f.
- We denote by $Ess(X)$ the set composed of all essential faces of X.

Note: Any principal face is essential.



The 2-face is not essential

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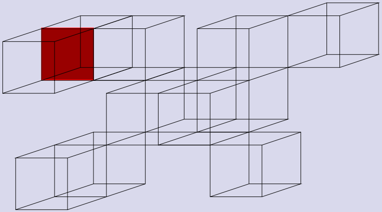

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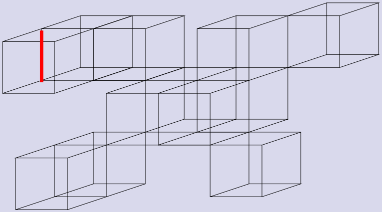

The 1-face is not essential

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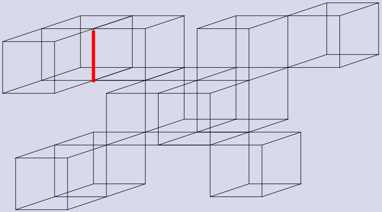

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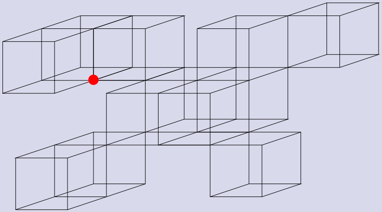

The 0-face is not essential

Essential face

Definition

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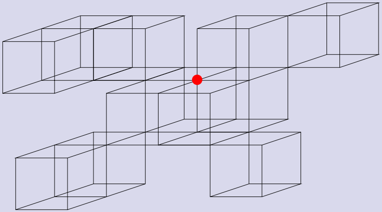
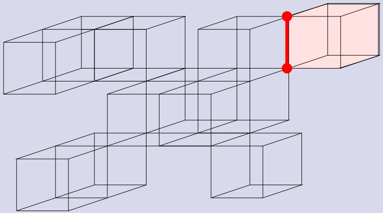


The 0-face is essential

Core

Definition

- The core of f is the complex, denoted by $Core(f, X)$, composed by all the essential faces which are strictly included in f, and all the faces included in these faces.



A 3-face and its core

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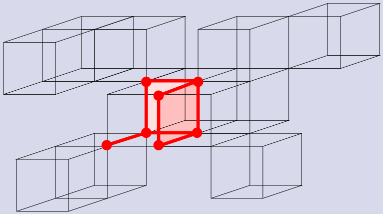

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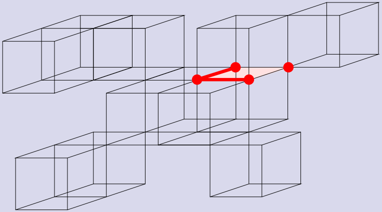
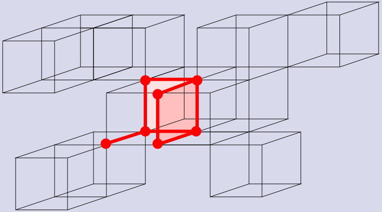


A 2-face and its core

Regular/critical face

Definition

- We say that f is regular if $f \in Ess(X)$ and if $\hat{f}$ collapses onto $Core(f, X)$.
- We say that f is critical if $f \in Ess(X)$ and if f is not regular.



The 3-face is regular

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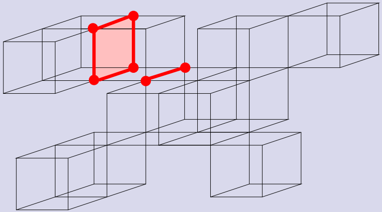

The 3-face is critical

Regular/critical face

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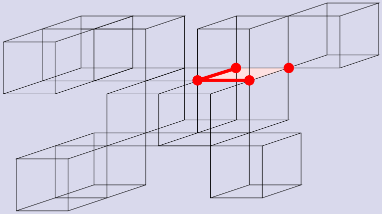


The 2-face is critical

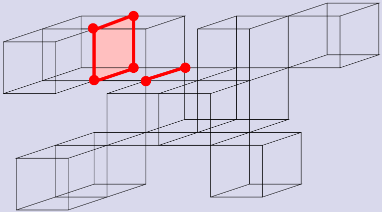
Local characterization of simple faces

Property

A principal face of a complex X is simple if and only if it is regular.

Note

We retrieve a local characterization established by T.Y. Kong, based on the notion of attachment.



The 3-face is not simple

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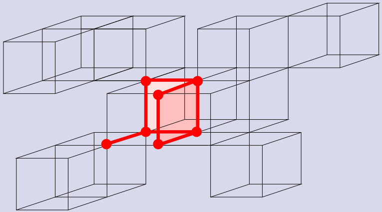
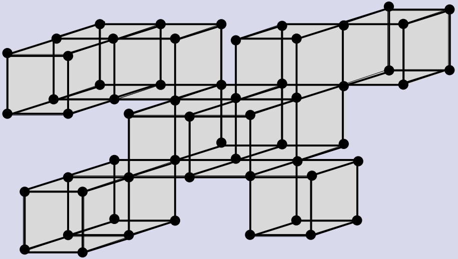


The 3-face is simple

Critical kernel

Definition

- We set $Critic(X) = \cup\{\hat{f} \mid f \text{ is critical }\}$, $Critic(X)$ is the critical kernel of X.



A complex X

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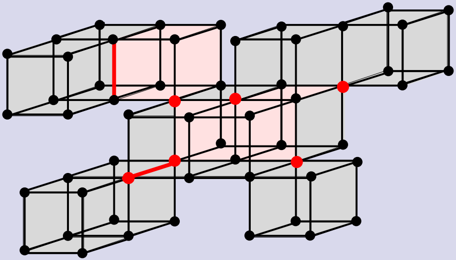

The critical faces of X

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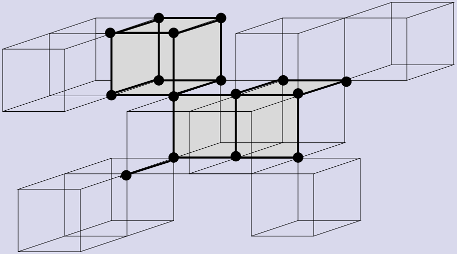
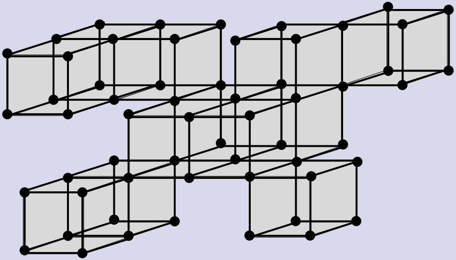


The critical kernel of X

Main theorem (1)

Theorem

In any dimension, the critical kernel of X is a retraction of X.



A complex X

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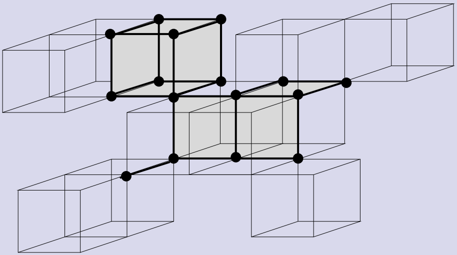

$X_1 = Critic(X)$, a retraction of X

Main theorem (1)

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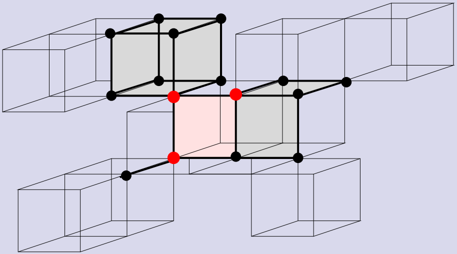

The critical faces of X_1

Main theorem (1)

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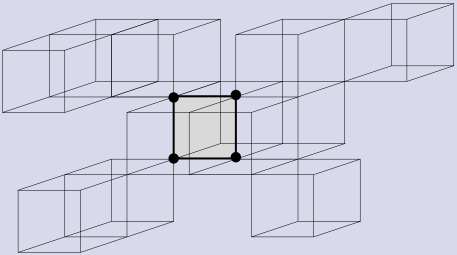


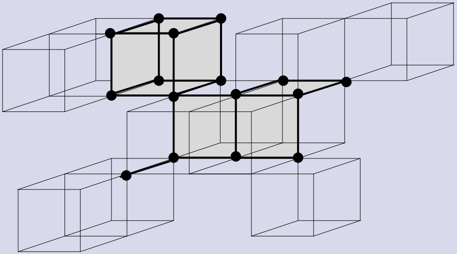
$X_2 = Critic(X_1)$, a retraction of X_1 such that $Critic(X_2) = X_2$

Main theorem (2)

Theorem

In any dimension, the critical kernel of X is a retraction of X.

Furthermore, if Y is any principal subcomplex of X such that Y contains the critical kernel of X, then Y is a retraction of X.



$Critic(X)$

Main theorem (2)

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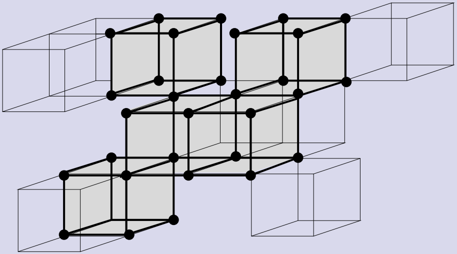

A complex Y which contains $Critic(X)$

Main theorem (2)

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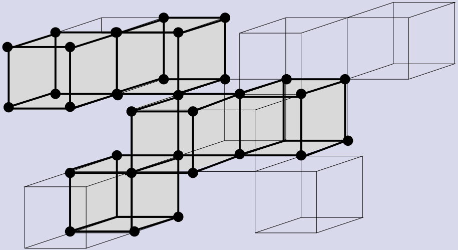


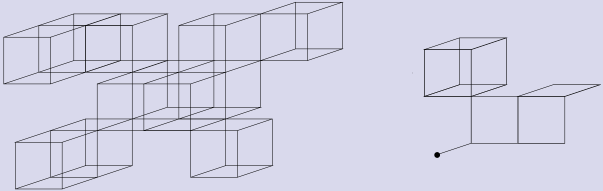
Another complex Z which contains $Critic(X)$

Part III

New 3D Parallel Thinning Scheme

Crucial kernels: motivation

The critical kernel of a set of voxels is not always a set of voxels

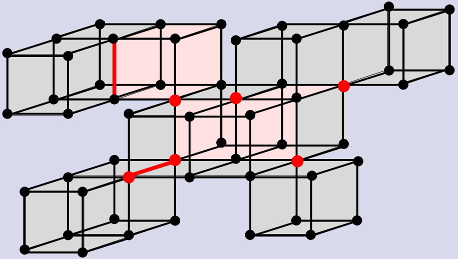


In the following, we assume that X is a set of voxels (*i.e.*, a complex in which each principal face is a 3-face).

Maximal critical face

Definition

- A face f in X is a maximal critical face, or an M-critical face, if f is a critical face which is not strictly included in any other critical face.



Critical faces

Maximal critical face

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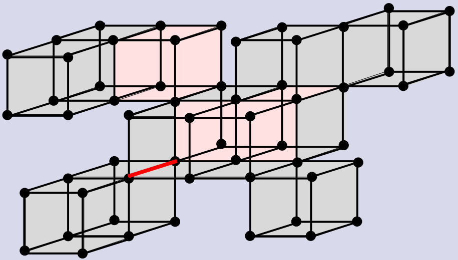

M-critical faces

Crucial kernel

Let K be a given subset of X. Loosely speaking,

- the crucial kernel of X constrained by K is a particular, uniquely defined set of voxels which contains all the M-critical faces of X and which contains K.

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Corollary of the main theorem

The following property is an easy consequence of the main theorem:

Property

Let K be a set composed of 3-faces of X. The crucial kernel of X constrained by K is a retraction of X.

Generic thinning scheme

The crucial kernel of X constrained by K is denoted by $Cruc(X, K)$.

Definition

Let $K \subseteq X$.

- Let $\langle X_0, X_1, ..., X_k \rangle$ be the unique sequence such that
  - $X_0 = X$, and
  - $X_k = Cruc(X_k, K)$, and
  - $X_i = Cruc(X_{i-1}, K)$, $i = 1, ..., k$.
- The set X_k is the $\mathcal{K}$-skeleton of X constrained by K.

Illustration: with $K = \emptyset$

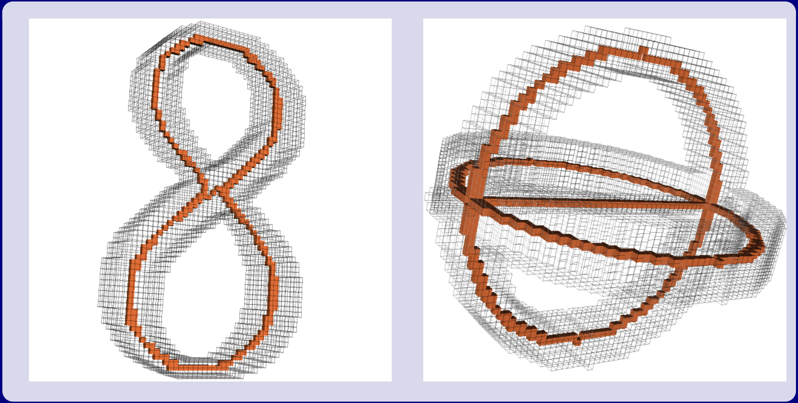


Illustration: with $K \subseteq AM(X)$

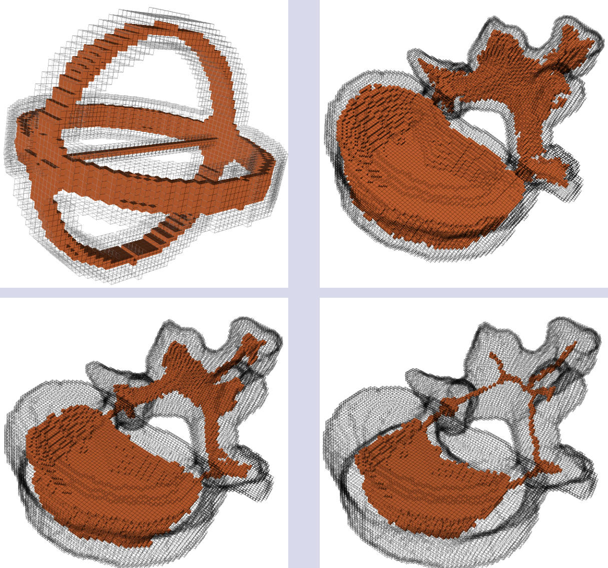
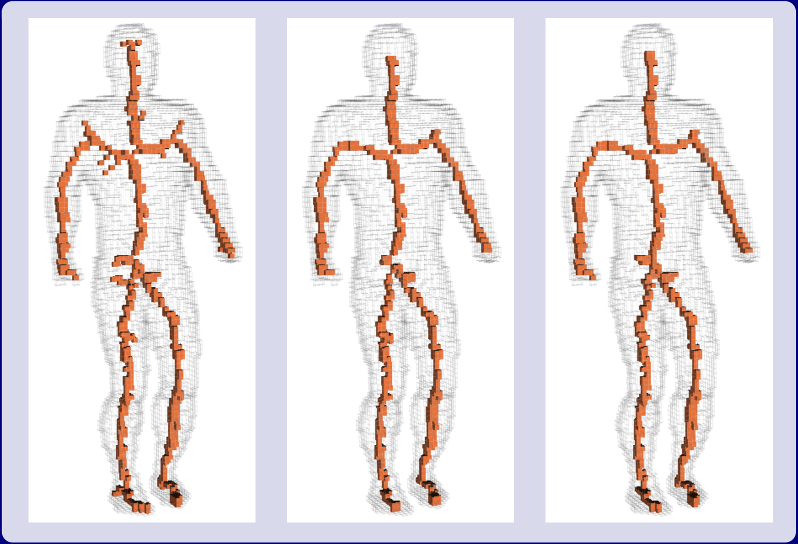


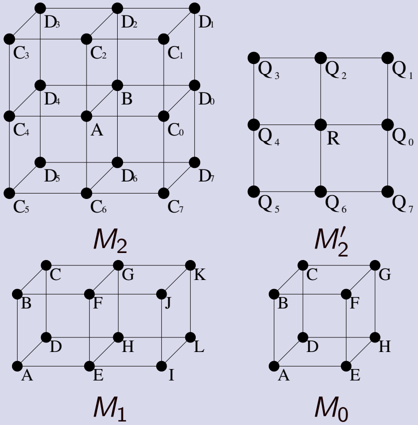
Illustration: curvilinear skeletons



Part IV

Local characterizations and thinning algorithm

Local conditions (3D)



Masks for 2-crucial (M_2), 1-crucial (M_1), and 0-crucial (M_0) voxels (3-faces), M_2' is a configuration derived from M_2. Here, a voxel is represented by a point.

Thinning algorithm

Algorithm SK^3 (Input: X, $K \subseteq X$; Output: X)

01. Repeat Until Stability
02. R_3 := set of voxels which are critical for X or which are in K
03. R_2 := set of voxels which are 2-crucial and included in $X \setminus R_3$
04. R_1 := set of voxels which are 1-crucial and included in $X \setminus (R_3 \cup R_2)$
05. R_0 := set of voxels which are 0-crucial and included in $X \setminus (R_3 \cup R_2 \cup R_1)$
06. $X := R_3 \cup R_2 \cup R_1 \cup R_0$

Part V

Epilogue

Conclusion

The salient outcomes of this work are:

- the definition and local characterization of 3D crucial faces, allowing for *fast and simple implementations*,
- a *new generic parallel 3D thinning scheme*,
- a parallel algorithm for a *minimal symmetric constrained skeleton*

As far as we know, all previously proposed symmetric thinning conditions are not sufficiently “powerful” for removing enough points in order to obtain a skeleton such as the minimal skeleton presented in this talk.

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Perspectives

- general skeletons (*i.e.*, which are not necessarily principal subcomplexes)
- link with P-simple points
- link with minimal non-simple sets
- N-dimensional thinning

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- link with minimal non-simple sets
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Perspectives

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Questions