New characterizations of simple points, minimal non-simple sets and P-simple points in 2D, 3D and 4D discrete spaces

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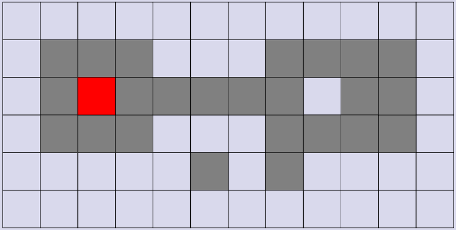
April 18, 2008

Homotopic thinning

Simple point

Intuitively, a point is simple if it can be removed without changing topology



non simple

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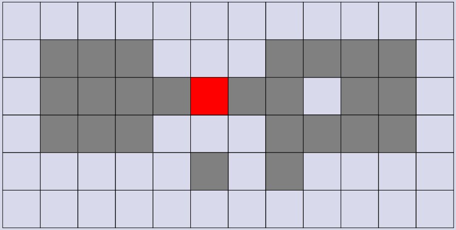

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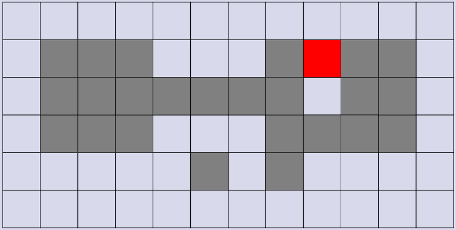

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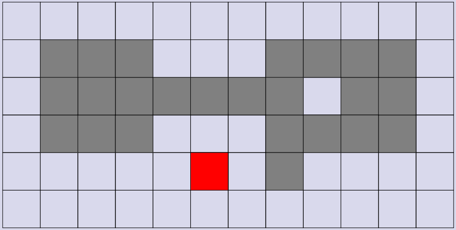

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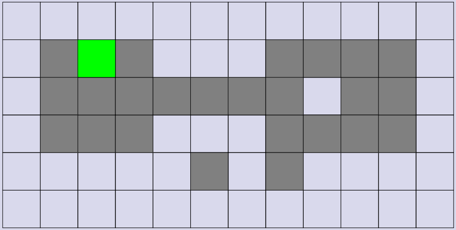

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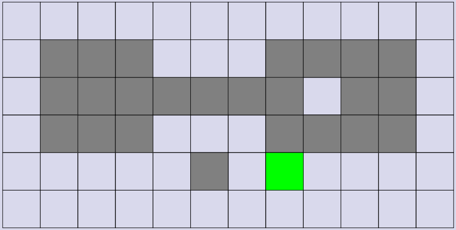

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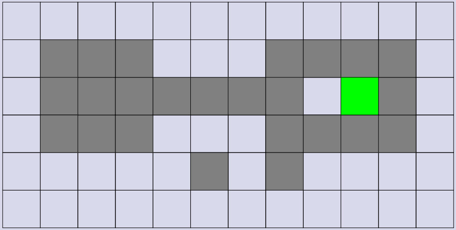
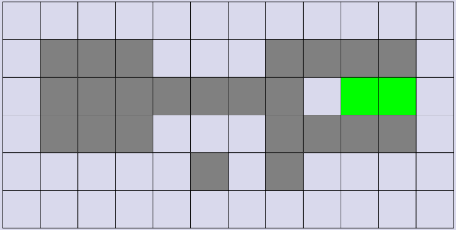


simple

Warning: parallel removal of simple points . . .

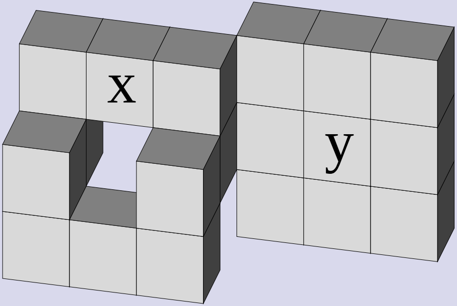
. . . may alter topology.



The 3D case

In 2D, the notion of connectedness (for both the object and the background) suffices to characterize simple pixels.

In 3D, things are more difficult.



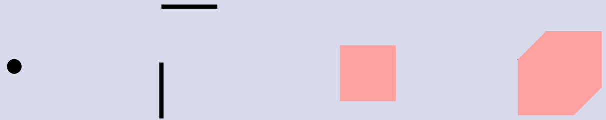
Plan of the presentation

- Cubical complexes, collapse and simple points
- Confluences and new characterizations of simple points in 2D, 3D, 4D
- Critical Kernels (parallel thinning)
- New characterizations of MNSs and P-simple points in 2D, 3D, 4D

Cubical complexes, collapse, simple points

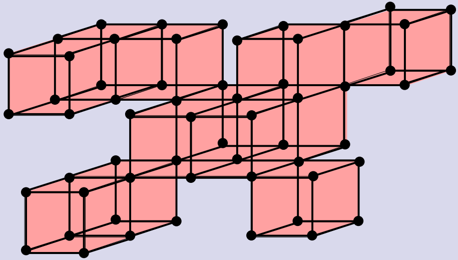
Cubical complexes

A complex is a set of faces glued together according to certain rules.



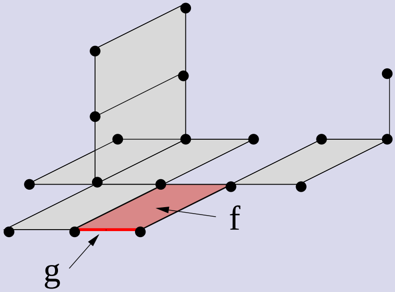
0-face | 1-faces | 2-face (pixel) | 3-face (voxel)

A complex:



Elementary collapse

- Let f and g be two distinct faces such that f is the only face of X which contains g.
- The complex $X \setminus \{f, g\}$ is an elementary collapse of X.



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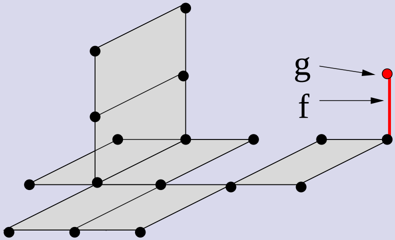

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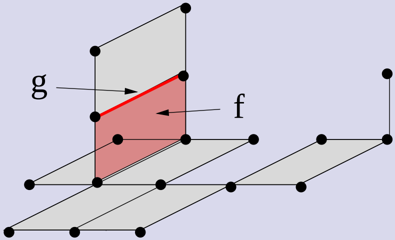
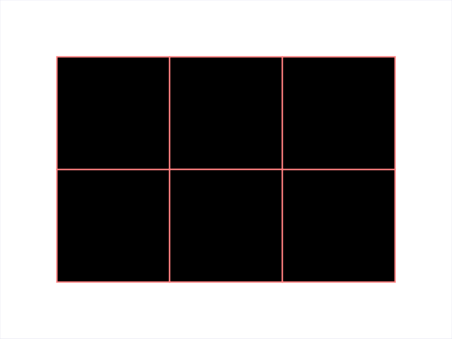


$X \setminus \{f, g\}$ is not an elementary collapse of X.

Collapse preserves topology



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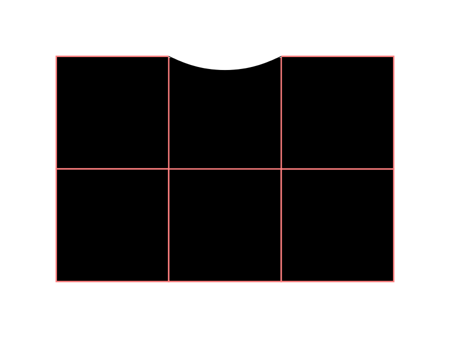

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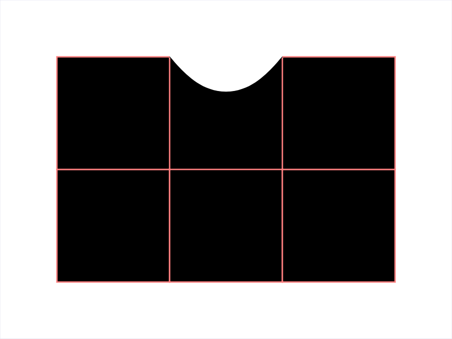

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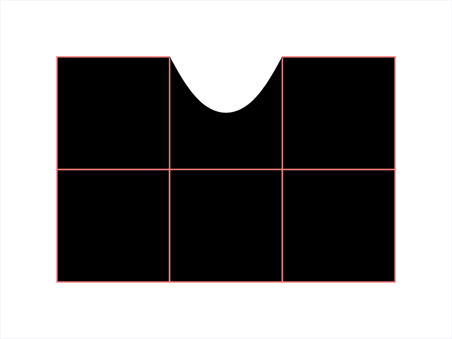

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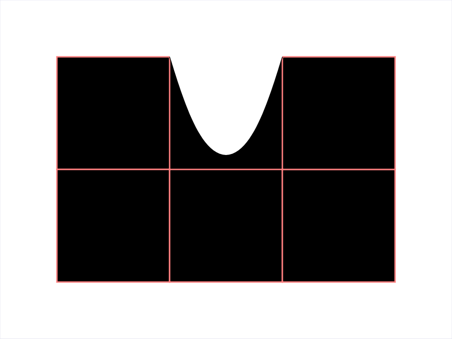

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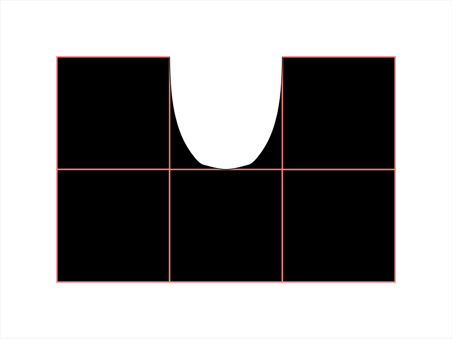

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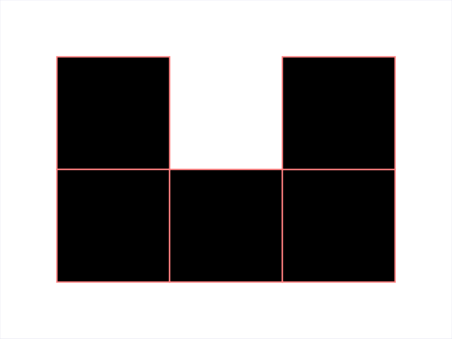
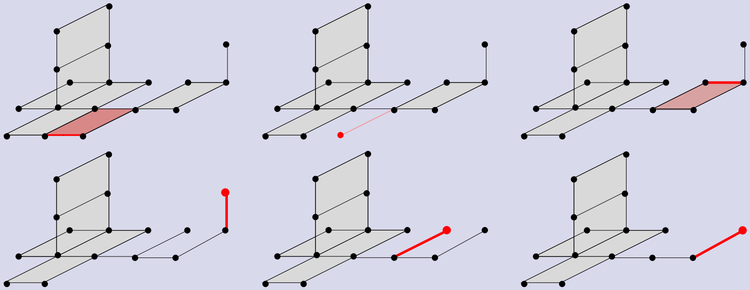


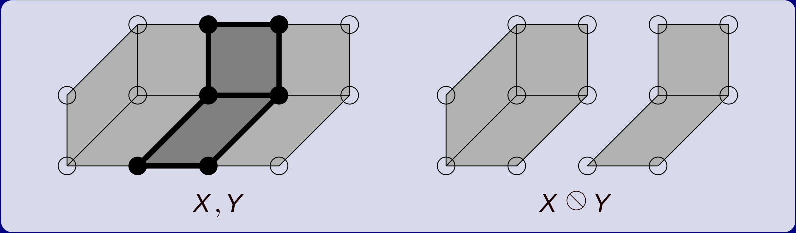
Collapse sequence

- Let X, Y be two complexes. We say that X collapses onto Y if there exists a collapse sequence from X to Y.



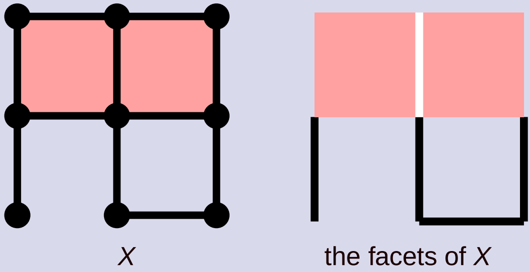
Detachment

The detachment operation (denoted by $\oslash$) "removes" a subcomplex from a complex, yielding a new complex.



Facet

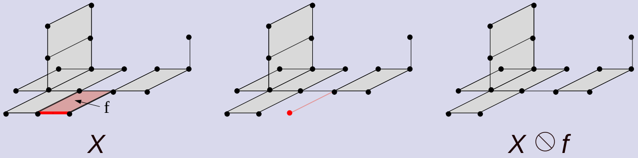
A face $f \in X$ is a facet if there is no $g \in X$ such that f is strictly included in g.



Definition of simple facets (G. Bertrand)

Definition

Let f be a facet, we say that f is simple if X collapses onto $X \oslash f$.

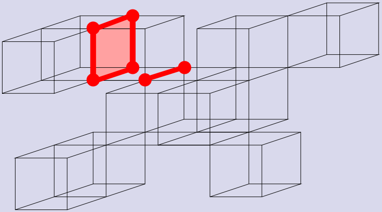


The facet f is simple.

Local characterization of simple facets

Let f be a facet of X. The attachment of f for X is the complex defined by $Att(f,X) = f \cap (X \oslash f)$.

Theorem: the facet f is simple for X if and only if f collapses onto $Att(f,X)$.



The facet is not simple

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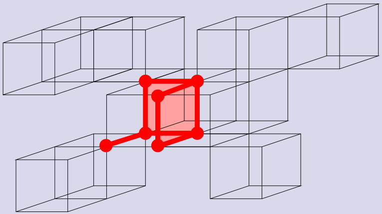
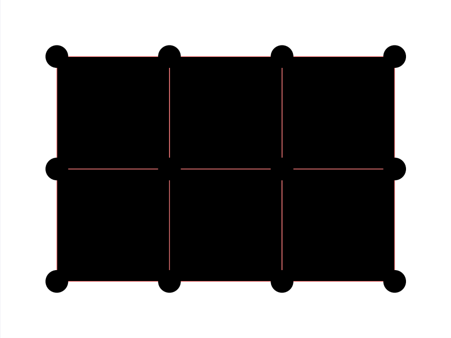


The facet is simple

Confluences and new characterizations of simple points in 2D, 3D, 4D

In 2D: a rectangle collapses onto a point

Collapsing a rectangle in any arbitrary order leads to a single point



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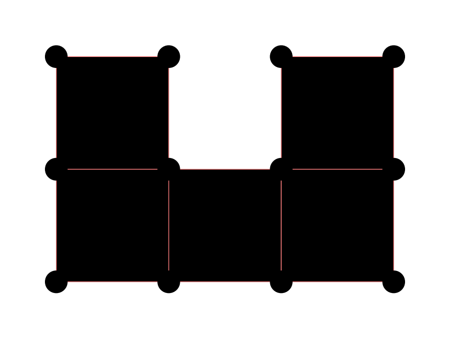

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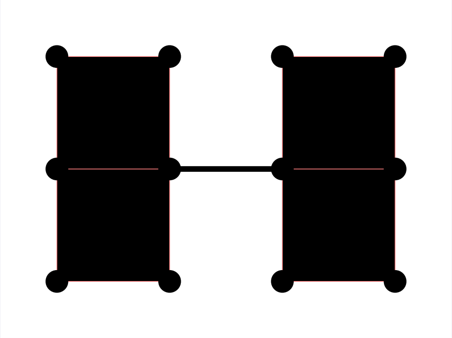

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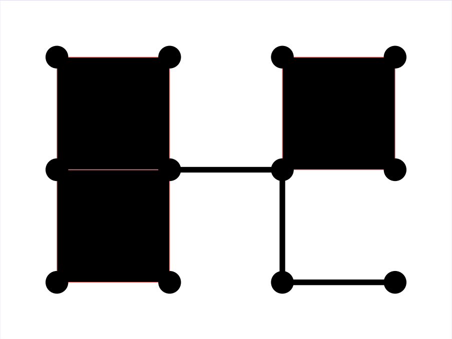

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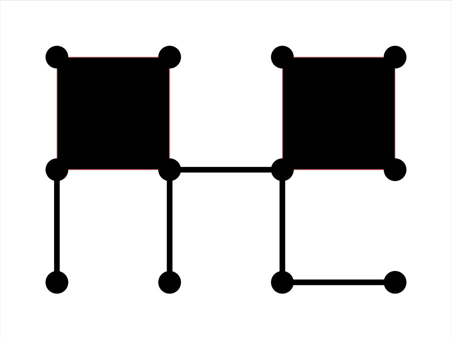

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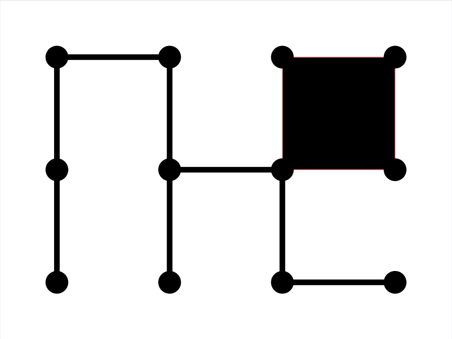

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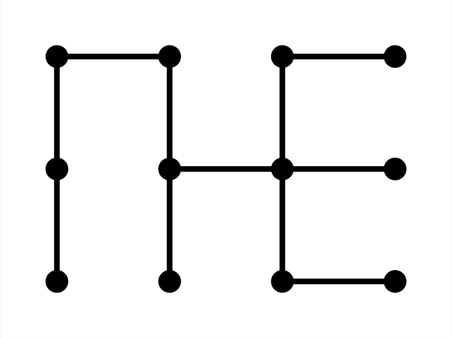

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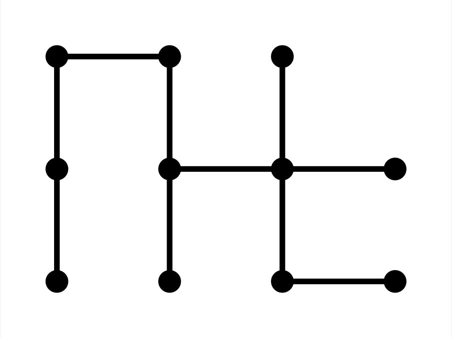

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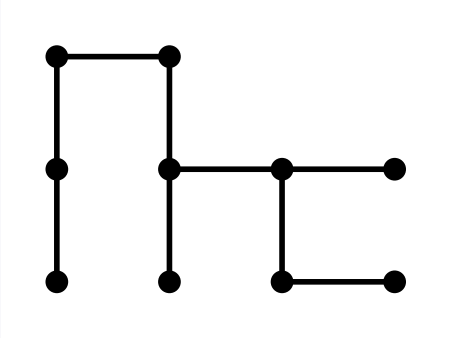

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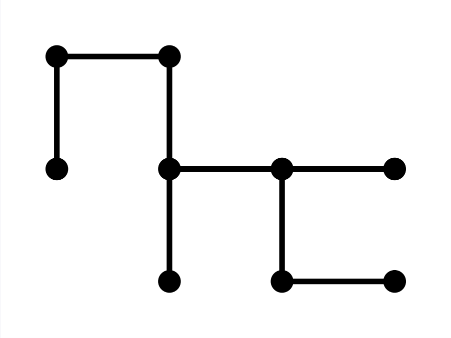

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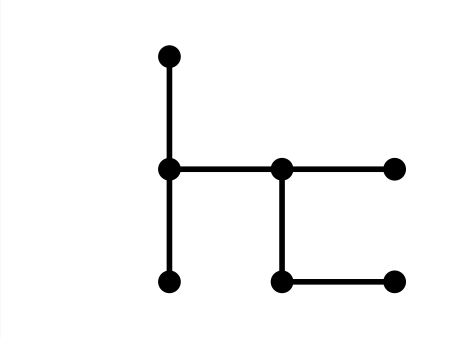

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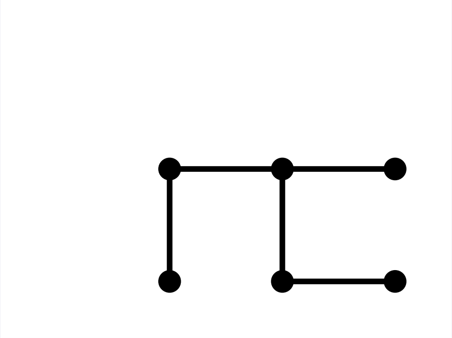

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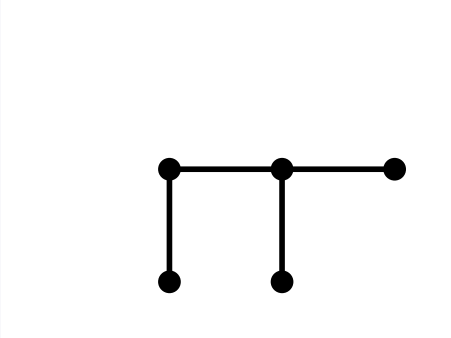

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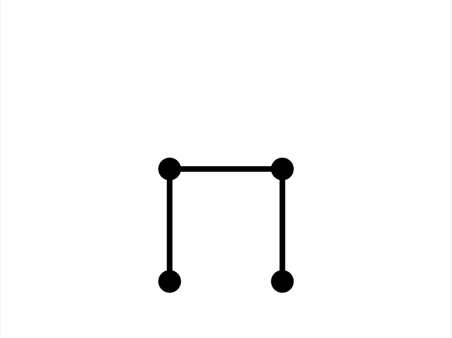

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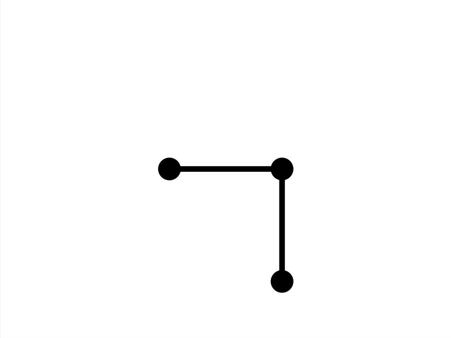

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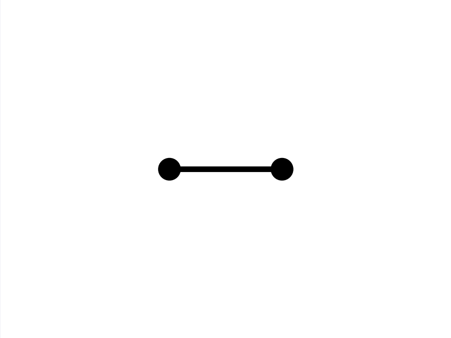

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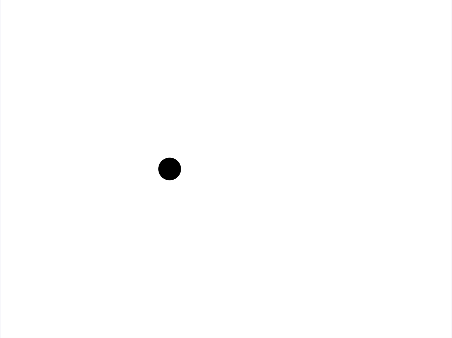
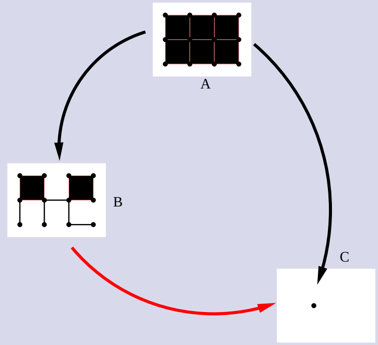


Confluence properties

Let A, B, C be three complexes such that $C \subset B \subset A$



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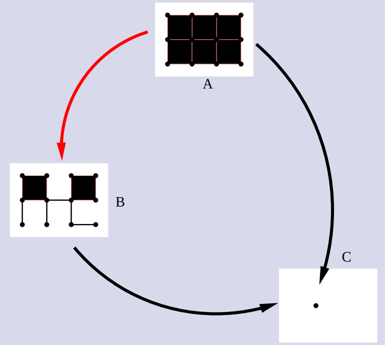
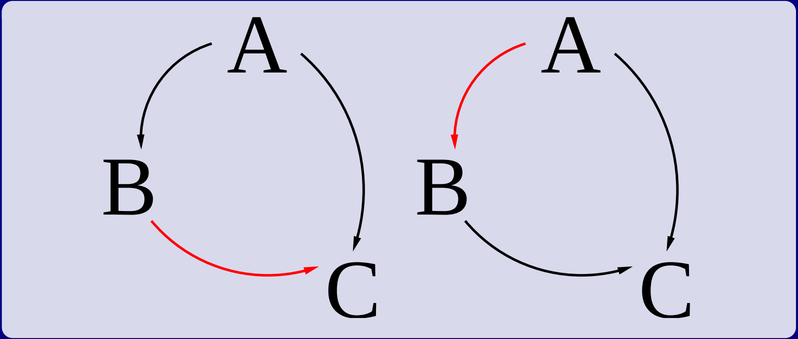


Confluence properties in 2D

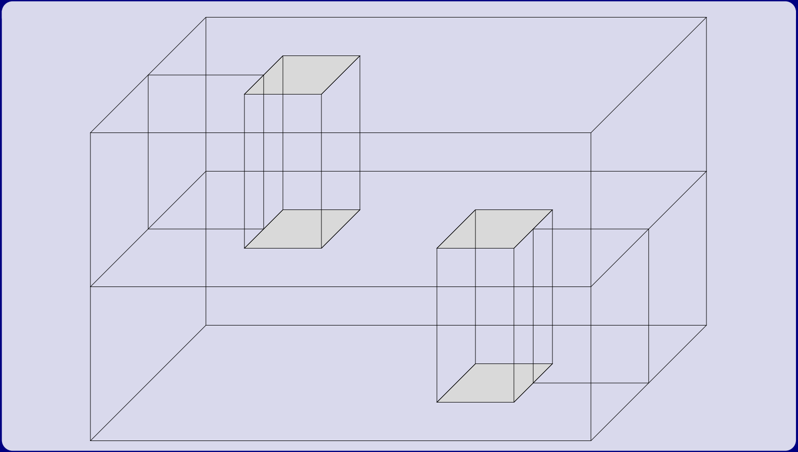
Let A, B, C be any three complexes in the 2-dimensionnal discrete space such that $C \subset B \subset A$.
The two following confluence properties hold:



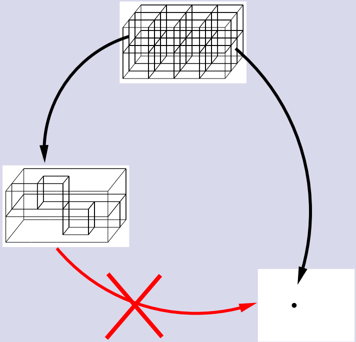
Question:

Are these confluence properties also true in 3D ?

Bing's house



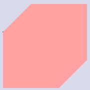
Bing's house is a counter-example for 3D confluence



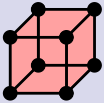
Bing's house has no free face.

New confluence properties

Let $d \in \{2, 3, 4\}$.
Let f be a d-face.
Let $\hat{f}$ be the complex formed by f and all the faces herein.



a 3-face f



the complex $\hat{f}$

New confluence properties

Let $d \in \{2, 3, 4\}$.
Let f be a d-face.
Let $\hat{f}$ be the complex formed by f and all the faces herein.

Theorem

Let A, B be subcomplexes of $\hat{f}$ such that $B \subset A$ and A collapses onto a point.
Then, B collapses onto a point if and only if A collapses onto B.

Theorem

Let C, D be subcomplexes of $\hat{f}$ such that $D \subset C$, and $\hat{f}$ collapses onto D.
Then, $\hat{f}$ collapses onto C if and only if C collapses onto D.

New characterization of simple points

Remind that:

The facet f is simple for X if and only if f collapses onto its attachment to X.

Applying this characterization may involve, in the worst case, the exploration of all possible collapse sequences, which is huge.

New characterization of simple points

Thanks to the above confluence properties, we proved that the following greedy collapsing algorithm allows for a characterization of simple facets

Algorithm $\mathcal{A}_1$: *Set* $Z = \hat{f}$ *; Set* $A = Att(f, X)$ *; Do*
Select any free pair (h, g) *in* $Z \setminus A$*; set* Z *to* $Z \setminus \{h, g\}$ *;*
Continue until either $Z = A$ *(answer yes) or no such pair is found (answer no).*

Critical kernels

A framework for the study of parallel thinning

Key notions

This framework, introduced by G. Bertrand for the study of parallel thinning in any dimension, is based on only three notions:

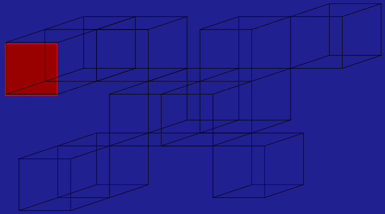
- Essential face
- Core of a face
- Regular/critical face

Essential face

Definition

We say that f is an essential face if f is precisely the intersection of all facets of X which contain f.

Note: Any facet is essential.



The 2-face is not essential

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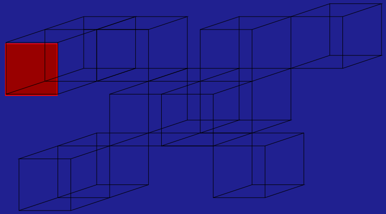

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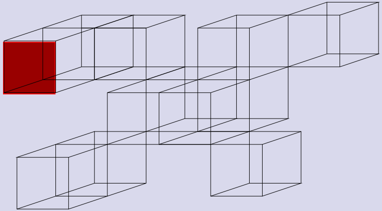

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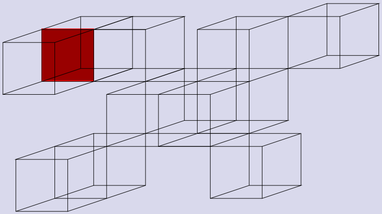

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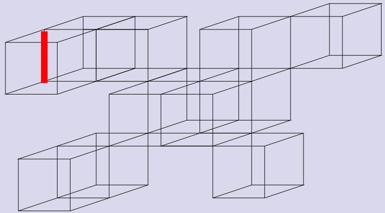

The 1-face is not essential

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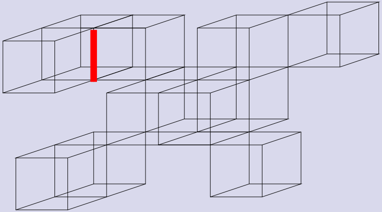

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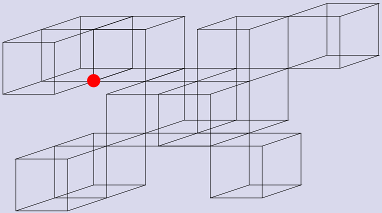

The 0-face is not essential

Essential face

Definition

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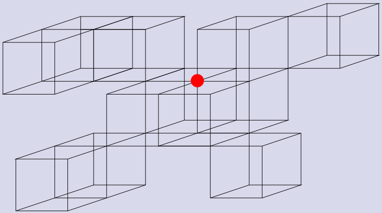
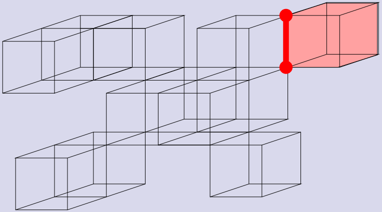


The 0-face is essential

Core

Definition

- The core of f is the complex, denoted by $Core(f, X)$, composed by all the essential faces which are strictly included in f, and all the faces included in these faces.



A 3-face and its core

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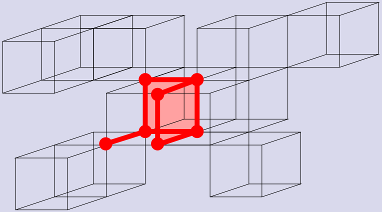

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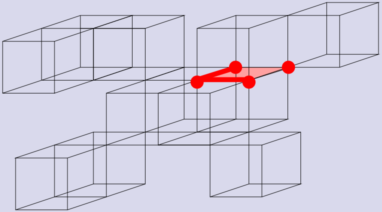
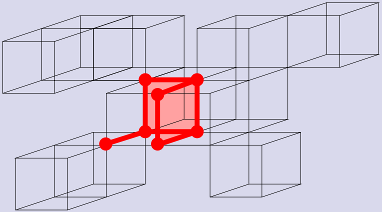


A 2-face and its core

Regular/critical face

Definition

- We say that f is regular if f is essential and if $\hat{f}$ collapses onto $Core(f, X)$.
- We say that f is critical if f is essential and not regular.



The 3-face is regular

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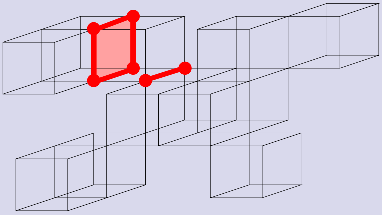

The 3-face is critical

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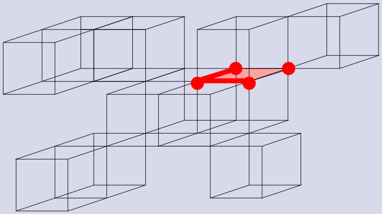
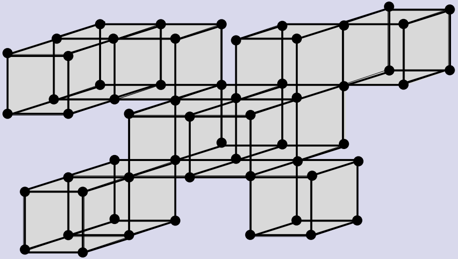


The 2-face is critical

Critical kernel

Definition

- We set $Critic(X) = \cup\{\hat{f} \mid f \text{ is critical }\}$, $Critic(X)$ is the critical kernel of X.



A complex X

Critical kernel

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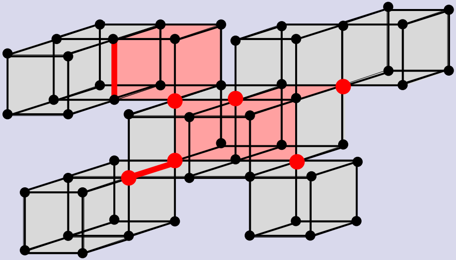

The critical faces of X

Critical kernel

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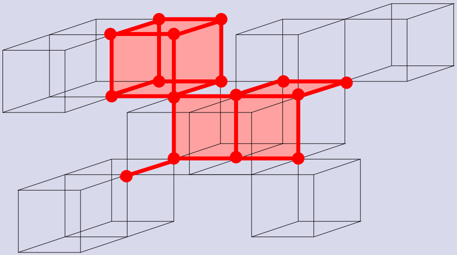


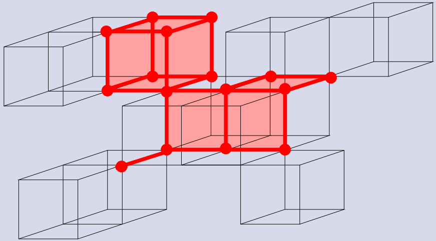
The critical kernel of X

Main theorem (G. Bertrand)

Theorem

In any dimension, X collapses onto the critical kernel of X.

Furthermore, if Y is any set of facets of X such that Y contains the critical kernel of X, then X collapses onto Y.



This theorem leads to a wide class of topologically correct *n*-D parallel thinning algorithms, based on the different possible choices of the set Y.

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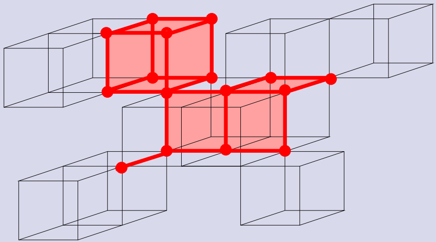
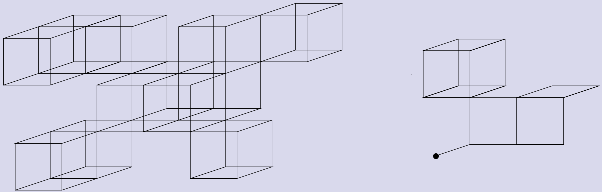


This theorem leads to a wide class of topologically correct n-D parallel thinning algorithms, based on the different possible choices of the set Y.

Crucial kernels: motivation

The critical kernel of a set of voxels is not always a set of voxels

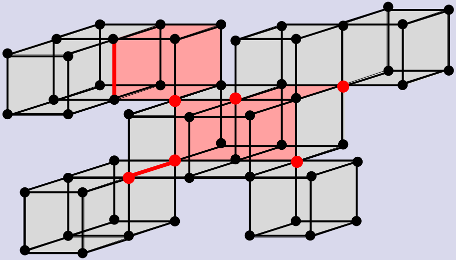


In the following, we assume that X is a set of voxels (*i.e.*, a complex in which each principal face is a 3-face).

Maximal critical face

Definition

- A face f in X is a maximal critical face, or an M-critical face, if f is a critical face which is not strictly included in any other critical face.



Critical faces

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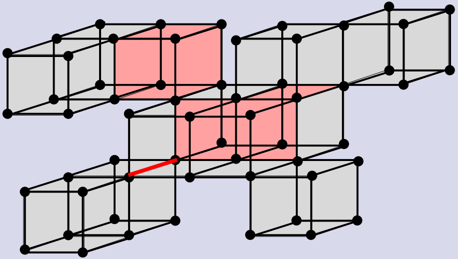


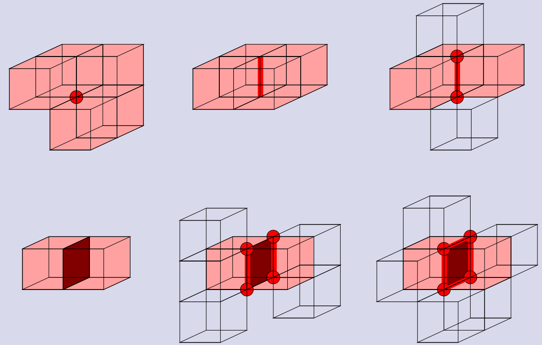
M-critical faces

Crucial cliques

Definition

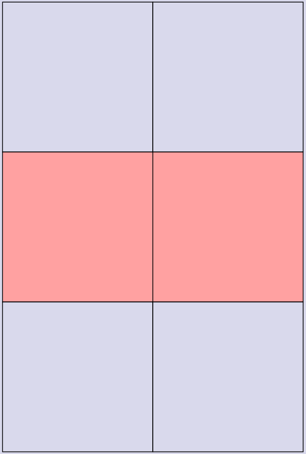
Let f be an M-critical face of X.
The set K of all the facets of X which contain f is called a crucial clique (for X)

Crucial cliques: examples

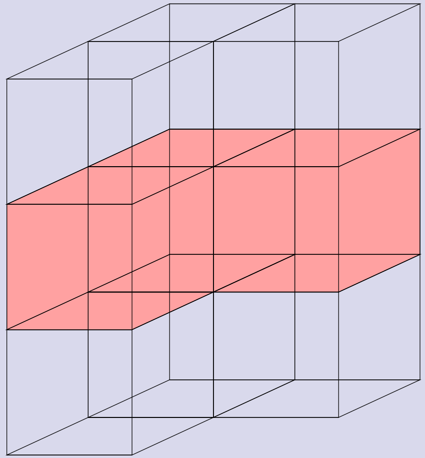


Minimal Non-simple Sets (MNS)

Minimal Non-simple Set, or MNS [Ronse 1988]



Minimal Non-simple Set, or MNS [Ronse 1988]



Extended by Kong, Gau to 3D and 4D

Minimal Non-simple Set, or MNS [Ronse 1988]

Any thinning algorithm which preserves at least one element of every MNS at each step is guaranteed to preserve topology.

New characterization of MNSs

Let X be a pure d-complex, with $d \in \{2, 3, 4\}$, and let K be a set of facets of X.

Theorem

The set K is a minimal non-simple set for X if and only if it is a crucial clique for X.

Conclusion

- Confluence properties (true up to 4D)
- New characterization of simple points, up to 4D
- New efficient simplicity testing algorithm
- MNSs are particular cases in the critical kernels framework
- P-simple points are particular cases in the critical kernels framework
- New efficient algorithms for detecting MNSs and P-simple points
- Non-combinatorial proofs (except for one lemma)

Higher dimensions...

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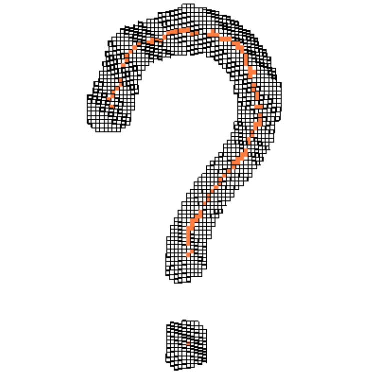
Higher dimensions...

Questions



Minimal Non-simple Set, or MNS [Ronse 1988]

A sequence $\langle k_0, \ldots, k_\ell \rangle$ of facets of X is said to be a simple sequence for X if k_0 is simple for X, and if, for any $i \in \{1, \ldots, \ell\}$, k_i is simple for $X \oslash \{k_j\ ,\ 0 \leq j < i\}$.

Let K be a set of facets of X. The set K is said to be F-simple for X if K is empty, or if the elements of K can be ordered as a simple sequence for X.

The set K is minimal non-simple for X if it is not F-simple for X and if all its proper subsets are F-simple.

P-simple points [Bertrand 1995]

Let C be a set of facets of X. A facet $k \in C$ is said to be P-simple for $\langle X, C \rangle$ if k is simple for all complexes $X \oslash T$, such that $T \subseteq C \setminus \{k\}$.

New characterization of P-simple points

Let X be a pure d-complex, with $d \in \{2, 3, 4\}$, and let C be a set of facets of X.

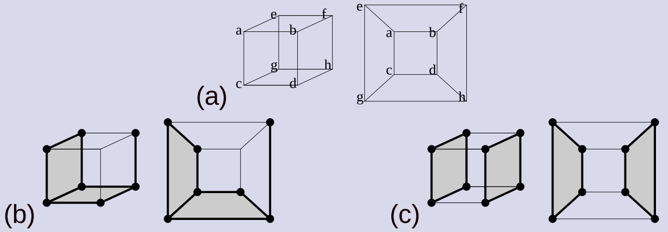
Theorem

A facet k in C is P-simple for $\langle X, C \rangle$ if and only if every face of k that is critical for X is also a face of a facet of X that is not in C.

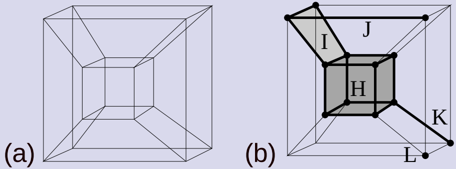
Higher dimensions (5D, 6D, . . . nD)

Part

Schlegel diagram



Schlegel diagram



A 1−subcomplex of the boundary of a 4−face

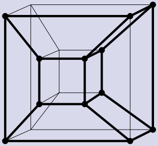
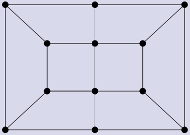
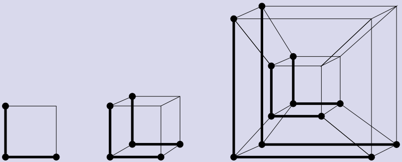
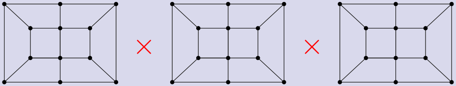
(a)  (b) 

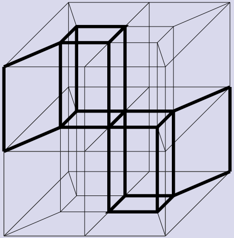
Illustration of the product operation



Sketch of a Bing's house in the boundary of a 6−face



CONTAINS



Counter-example in 5D

Computer program:

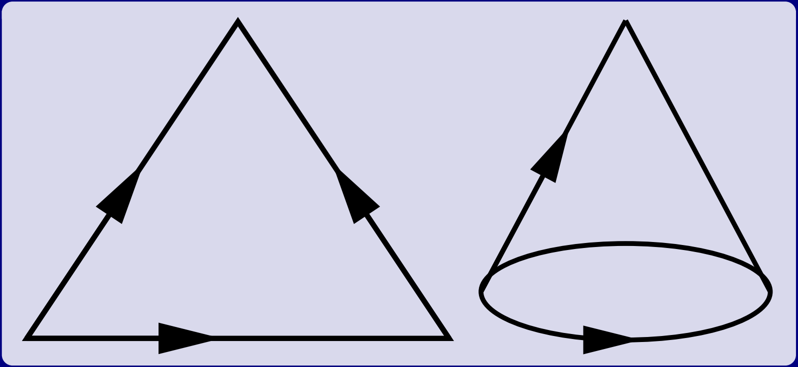
Loop forever

Randomly collapse a 5-face until stability

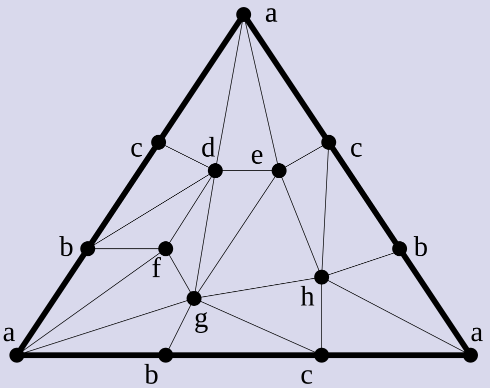
If the result X is not a singleton, then return X

Michel Couprie and Gilles Bertrand

Dunce hat

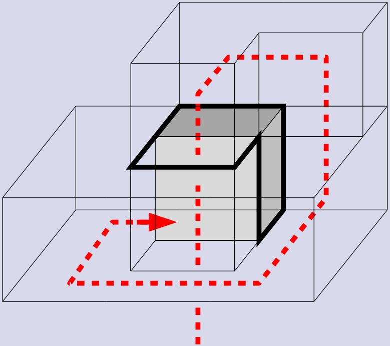


Dunce hat

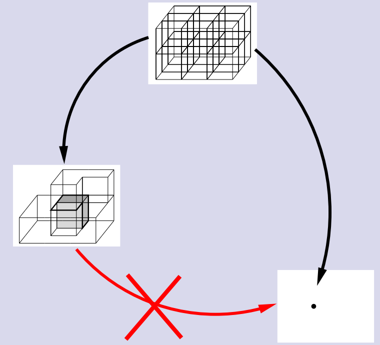


Dunce hat as a 2D complex

A cuboid collapses onto the Dunce hat



Dunce hat is a counter-example for 3D confluence



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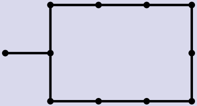
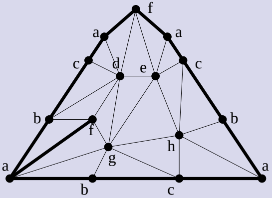


(a)



(b)

(a): The signature of X_{105}.

(b): A variant of the dunce hat (triangulated).