New 2D parallel thinning algorithms based on critical kernels

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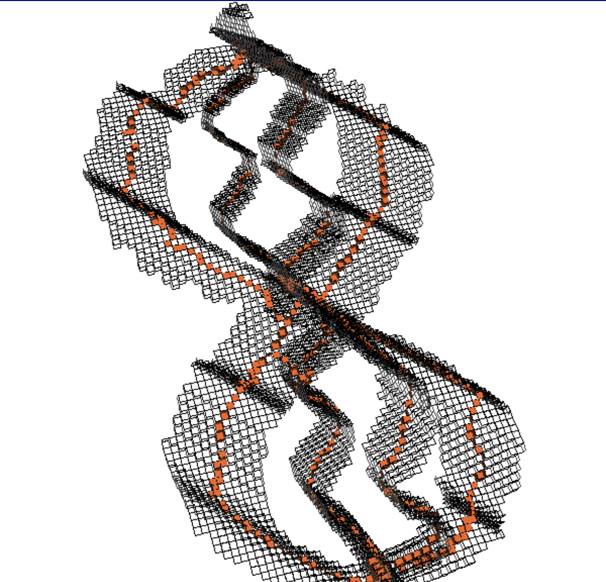
Parallel 2D thinning



Milestones

- 1966: D. Rutovitz – first parallel thinning algorithm
- 1970: A. Rosenfeld – digital topology
- 1988: C. Ronse – minimal non-simple sets
- 1995: G. Bertrand – P-simple points
- 2005: G. Bertrand – Critical kernels

Plan of the presentation

- Prologue
- 2D Cubical complexes (in nD grids)
- Critical kernels
- Crucial kernels
- $\mathcal{K}$-Skeletons in 2D grids
- $\mathcal{K}$-Skeletons of 2D objects in 3D grids
- Epilogue

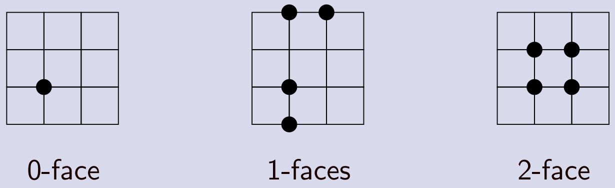
Part I

Cubical complexes

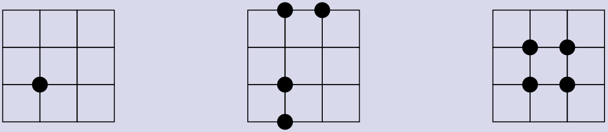
Face

We consider $\mathbb{Z}^n$, $n \geq 2$.

- A subset of $\mathbb{Z}^n$ composed of one point is called a 0-face.
- A subset of $\mathbb{Z}^n$ forming a unit bipoint is called a 1-face.
- A subset of $\mathbb{Z}^n$ forming a unit square is called a 2-face.

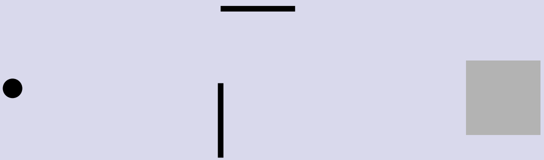


Faces: graphical representations



0-face 1-faces 2-face

Graphical representations:

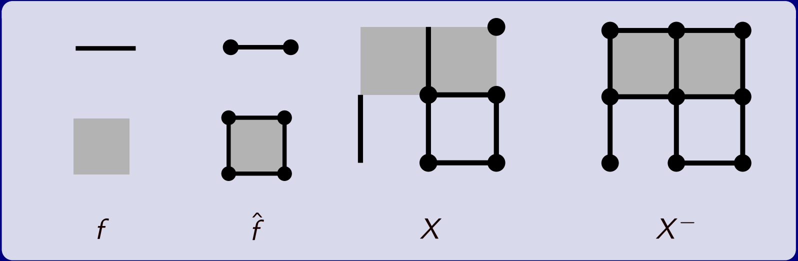


0-face 1-faces 2-face

Closure

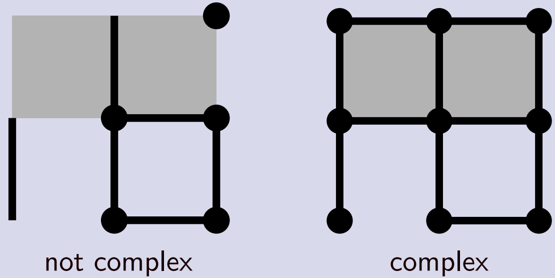
Let f be a face.

- The closure of f, denoted by $\hat{f}$, is the set composed by all the faces which are included in f.
- The set $\hat{f}$ is called a cell.
- If X is a finite set of faces, we write $X^- = \cup\{\hat{f} \mid f \in X\}$, X^- is the closure of X.



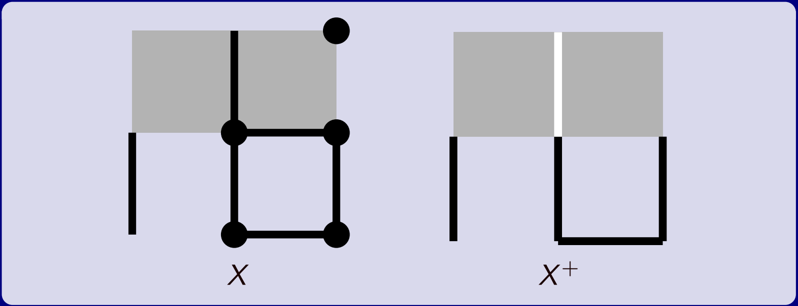
Cubical complex

A finite set X of faces is a complex if $X = X^-$.



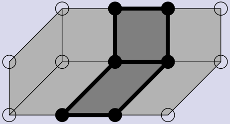
Principal face

- A face $f \in X$ is principal if there is no $g \in X$ such that f is strictly included in g.
- We denote by X^+ the set composed of all principal faces of X.

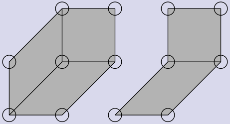


Detachment

Let Y be a subcomplex of X. We set $X \oslash Y = [X^+ \setminus Y^+]^-$. The set $X \oslash Y$ is a complex which is the detachment of Y from X.



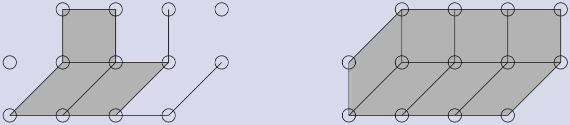
X, Y



$X \oslash Y$

Dimension, pure complex

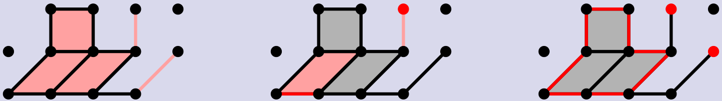
- Let X be a complex, $dim(X) = \max\{dim(f) \mid f \in X^+\}$ is the dimension of X.
- We say that X is an m-complex if $dim(X) = m$.
- We say that X is pure if, for each principal face f of X, we have $dim(f) = dim(X)$.



a non-pure 2-complex a pure 2-complex

Border face, free face

- The face f is a border face if there exists one face $g \in \hat{f}$, $g \neq f$, such that f is the only face of X which contains g.
- Such a face g is said to be free and the pair (f, g) is said to be a free pair.

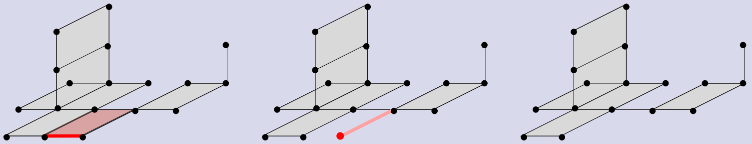


border faces — two free pairs — free faces

Collapse, retraction

- Let (f, g) be a free pair. The complex $X \setminus \{f, g\}$ is an elementary collapse of X.
- Let X, Y be two complexes. We say that X collapses onto Y if there exists a collapse sequence from X to Y.
- If X collapses onto Y, we also say that Y is a retraction of X.

Important: collapse preserves topology



A 2-step collapse sequence

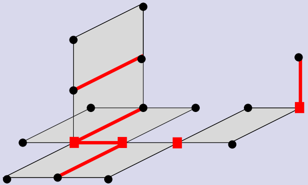
Part II

Critical kernels

Essential face

Definition

- We say that f is an essential face, or that $\hat{f}$ is an essential cell, if f is precisely the intersection of all principal faces of X which contain f.
- We denote by $Ess(X)$ the set composed of all essential faces of X.

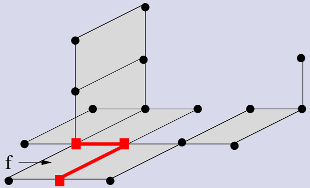


The essential 0- and 1-faces are highlighted.
All the 2-faces are principal, thus they are essential.

Core

Definition

- The core of $\hat{f}$ is the complex, denoted by $Core(\hat{f}, X)$, which is the union of all essential cells which are strictly included in f.

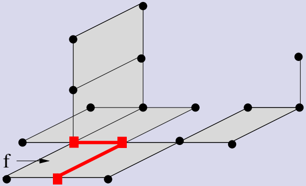


core of f core of g

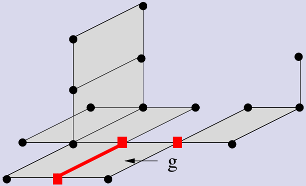
Critical face

Definition

- We say that f and $\hat{f}$ are regular if $f \in Ess(X)$ and if $\hat{f}$ collapses onto $Core(\hat{f}, X)$.
- We say that f and $\hat{f}$ are critical if $f \in Ess(X)$ and if f is not regular.



f is regular

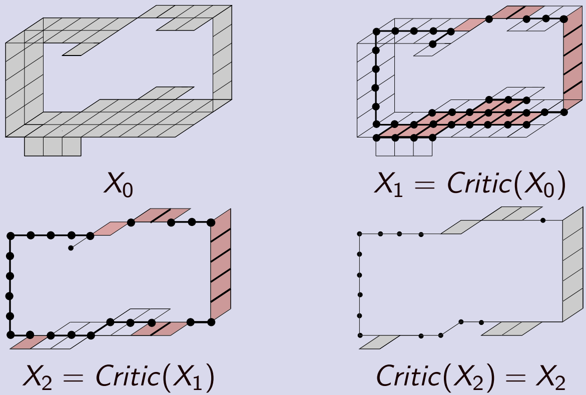


g is critical

Critical kernel

Definition

- We set $Critic(X) = \cup\{\hat{f} \mid f \text{ is critical }\}$, $Critic(X)$ is the critical kernel of X.



Main theorem

Theorem (G. Bertrand)

In any dimension, the critical kernel of X is a retraction of X.

Furthermore, if Y is any principal subcomplex of X such that Y contains the critical kernel of X, then Y is a retraction of X.

Simple cell

Intuitively, a cell $\hat{f}$ of a complex X is simple if its removal from X "does not change the topology of X".

Definition

Let f be a principal face, we say that $\hat{f}$ is simple if X collapses onto $X \oslash \hat{f}$.

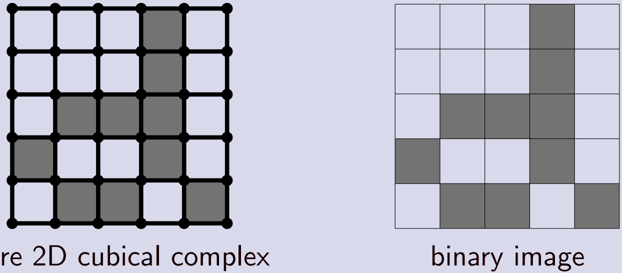
Property (local characterization)

A principal face of a complex X is simple if and only if is regular.

Part III

Crucial kernels

Motivation

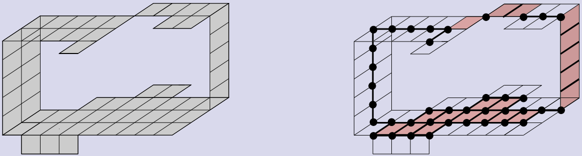


2-faces $\leftrightarrow$ pixels

Motivation

By identifying 2-faces with pixels in 2D, or surfels in 3D, we make a link between complexes and digital topology. But:

The critical kernel of a pure 2-complex is not always a pure 2-complex

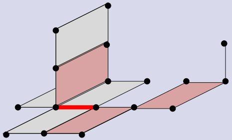


X_0 $X_1 = Critic(X_0)$

Maximal critical face

Definition

- A face f in X is a maximal critical face, or an M-critical face, if f is a critical face which is not strictly included in any other critical face.



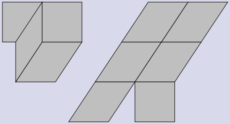
A complex X_0 and its M-critical faces (highlighted)

Crucial faces, crucial cliques

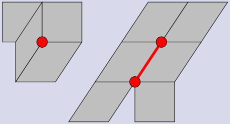
Definition

- We say that f is crucial, if $\hat{f} \setminus \{f\}$ contains a face which is M-critical.
- We say that f is 0-critical if $Core(\hat{f}, X) = \emptyset$;
  We say that f is 1-critical if $Core(\hat{f}, X)$ is not connected.
- We say that f is k-crucial if $\hat{f} \setminus \{f\}$ contains an M-critical face which is k-critical, $k = 0, 1$.
- We say that K is a (k-) crucial clique, if there exists a (k-critical) face f which is M-critical and such that K is precisely the set of principal faces of X which contain f.

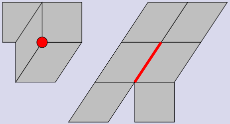
Crucial faces, crucial cliques



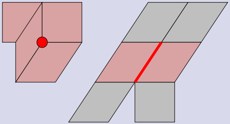
A complex



Critical faces



M-critical faces
left: 0-critical, right: 1-critical



Crucial cliques
left: 0-crucial, right: 1-crucial

We want a pure 2-complex which contains all 0- and 1-crucial faces. Thus, we have to preserve at least one face of each crucial clique.

Part IV

$\mathcal{K}$-Skeletons in 2D grids

Local conditions (2D)

A	A
P	P
B	B

C

0	P
P	0

C_1

0	0
P	P
0	0

C_2

	0	0
0	P	P
0	P	0

C_3

	0	0	
0	P	P	0
0	P	P	0
	0	0	

C_4

Property

Let S be a set of pixels and P be a set of simple pixels.

- *The pixel p is 1-crucial for $\langle S, P \rangle$ if and only if p is matched by pattern C;*
- *The pixel p is 0-crucial for $\langle S, P \rangle$ if and only if p is matched by one of the patterns $C_1, \ldots C_4$.*

Minimal $\mathcal{K}$-skeleton

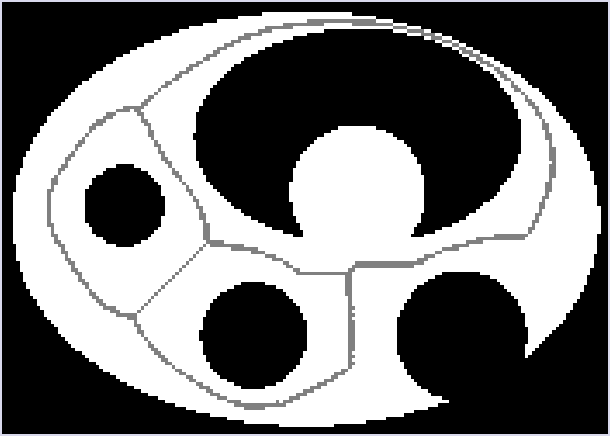
Definition

- The crucial kernel of S is the set $Cruc(S)$ which is composed of all critical pixels and all crucial pixels of S.
- Let $\langle S_0, S_1, ..., S_k \rangle$ be the unique sequence such that $S_0 = S$, $Cruc(S_k) = S_k$ and $S_i = Cruc(S_{i-1})$, $i = 1, ..., k$. The set S_k is the minimal $\mathcal{K}$-skeleton of S.

Algorithm MK_a^2 (**Input** /**Output** : a set S of pixels)

```
01. Repeat Until Stability
02.     P ← set of pixels which are simple for S
03.     R ← set of pixels in P which are 0- or 1-crucial for S
04.     S ← [S \ P] ∪ R
```

Minimal $\mathcal{K}$-skeleton: example



An object and its minimal $\mathcal{K}$-skeleton

Minimal $\mathcal{K}$-skeleton: 1-mask algorithm

Property

Let S be a set of pixels, and let $p \in S$ be a simple pixel.

- *If p is not crucial for S, then there exists $q \in \Gamma^*(p) \cap S$ such that q is either critical or 1-crucial for S.*
- *If p is 0-crucial for S, then any $q \in \Gamma^*(p) \cap S$ is neither critical, nor 1-crucial.*

Algorithm MK^2 (**Input** /**Output** : a set S of pixels)
01. **Repeat Until Stability**
02. $P \leftarrow$ set of pixels which are simple for S
03. $R \leftarrow$ set of pixels in P which are 1-crucial for S
04. $T \leftarrow [S \setminus P] \cup R$
05. $S \leftarrow T \cup [S \setminus (T \oplus \Gamma^*)]$

The correctness of the algorithm lies on the above property.

Constrained $\mathcal{K}$-skeleton

Definition

- Let S be a set of pixels and let $K \subseteq S$. Let P be the set composed of all simple pixels for S which are not in K.
- We denote by $Cruc(S, K)$ the set composed of all pixels in $S \setminus P$ and all pixels which are crucial for $\langle S, P \rangle$.
- Let $\langle S_0, S_1, ..., S_k \rangle$ be the unique sequence such that $S_0 = S$, $S_k = Cruc(S_k, K)$ and $S_i = Cruc(S_{i-1}, K)$, $i = 1, ..., k$.
- The set S_k is the $\mathcal{K}$-skeleton of S constrained by K.

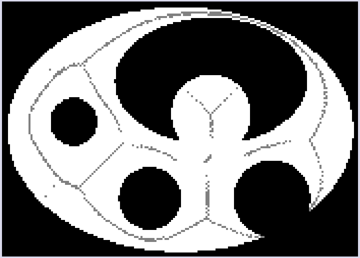
$\mathcal{K}$-Skeleton constrained by the medial axis

Algorithm AK^2 (**Input** /**Output** : set S of pixels)
00. $K \leftarrow \emptyset$; $T \leftarrow S$
01. **Repeat Until Stability**
02. $E \leftarrow T \ominus \Gamma_{\mathcal{S}}$; $D \leftarrow T \setminus [E \oplus \Gamma_{\mathcal{S}}]$
03. $T \leftarrow E$; $K \leftarrow K \cup D$
04. $P \leftarrow$ set of pixels of $S \setminus K$ which are simple for S
05. $R \leftarrow$ set of pixels in P which are 1-crucial for $\langle S, P \rangle$
06. $S \leftarrow [S \setminus P] \cup R$

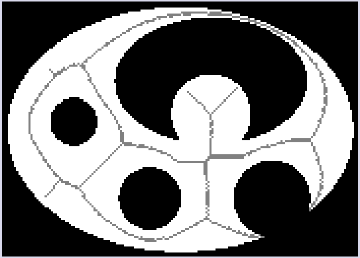
Property

Let S be a set of pixels. The set $AK^2(S)$ is the $\mathcal{K}$-skeleton of S constrained by the medial axis of S (relative to the 4-distance).

$\mathcal{K}$-Skeleton constrained by the medial axis



Medial axis

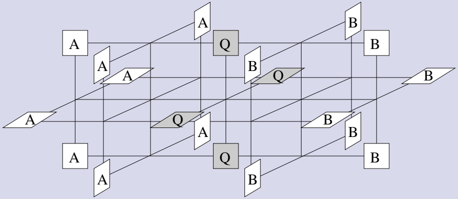


Constrained skeleton

Part V

$\mathcal{K}$-Skeletons of 2D objects in 3D grids

Local conditions (3D)



D

Property

Let S be a set of pixels and let $p \in S$. Let P be a set of simple surfels of S.

The surfel p is 1-crucial for $\langle S, P \rangle$ if and only if p is matched by the pattern D.

Minimal 2D $\mathcal{K}$-skeleton in the 3D grid

Algorithm MK_2^3 (**Input** /**Output** : a set S of surfels)

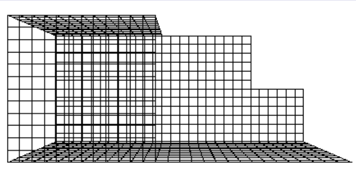
Repeat Until Stability

$P \leftarrow$ set of surfels which are simple for S

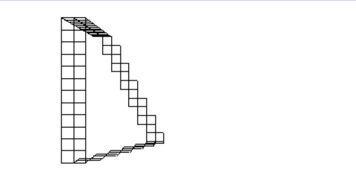
$R \leftarrow$ set of surfels in P which are 1-crucial for S

$T \leftarrow [S \setminus P] \cup R$

$S \leftarrow T \cup [S \setminus (T \oplus \Gamma^*)]$



Object S

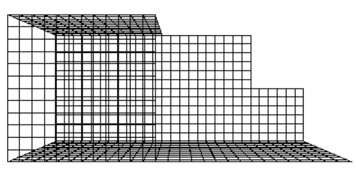


Minimal $\mathcal{K}$-skeleton of S

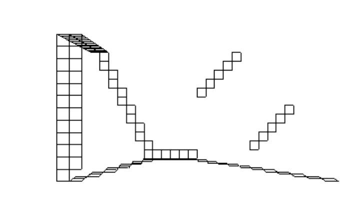
Topological axis: generalization of the medial axis

Definition

Let X be a complex, and let $f \in X^+$. We set $\rho(f, X)$ as the minimum length of a collapse sequence of X necessary to remove f from X, if such a sequence exists, and $\rho(f, X) = \infty$ otherwise. We define the topological axis of X as the set of faces f in X^+ such that $\rho(f, X) = \infty$ or $\rho(f, X) \geq \max\{\rho(g, X) \mid g \in \Gamma^*_{\mathcal{S}}(f) \text{ and } \rho(g, X) \neq \infty\}$.



Object



Topological axis

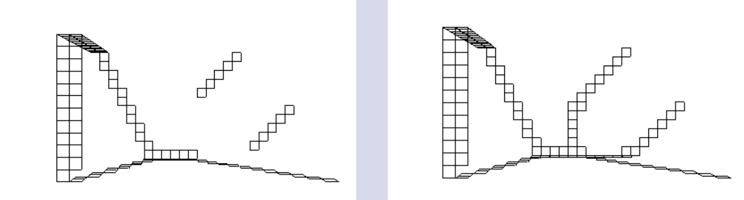
$\mathcal{K}$-Skeleton constrained by the topological axis

Algorithm BK_2^3 (**Input** /**Output** : set S of surfels)
00. $T \leftarrow S$
01. **Repeat Until Stability**
02. $T \leftarrow \{s \in T \mid s$ is an interior surfel of $T\}$
03. $P \leftarrow$ set of simple surfels for S such that $\Gamma_S^*(p) \cap T \neq \emptyset$
04. $R \leftarrow$ set of surfels in P which are 1-crucial for $\langle S, P \rangle$
05. $S \leftarrow [S \setminus P] \cup R$

Property

Let S be a set of pixels. The set $BK_2^3(S)$ is the $\mathcal{K}$-skeleton of S constrained by the topological axis of S.

$\mathcal{K}$-Skeleton constrained by the topological axis



Topological axis — Constrained skeleton

Part VI

Epilogue

Conclusion

The salient outcomes of this work are:

- the definition and some characterizations of crucial faces, allowing for *fast and simple implementations*,
- the definition and an algorithm for a *minimal symmetric skeleton* (MK^2),
- the introduction of the *topological axis*, which generalizes the medial axis,
- a parallel algorithm for a *symmetric skeleton which contains the medial axis*,
- a parallel algorithm for a *minimal symmetric skeleton of an object made of surfels*,
- a parallel algorithm for a *symmetric skeleton, which contains the topological axis* of an object made of surfels.

As far as we know, all the above algorithms have no equivalent.

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- the definition and some characterizations of crucial faces, allowing for *fast and simple implementations*,
- the definition and an algorithm for a *minimal symmetric skeleton* (MK^2),
- the introduction of the *topological axis*, which generalizes the medial axis,
- a parallel algorithm for a *symmetric skeleton which contains the medial axis*,
- a parallel algorithm for a *minimal symmetric skeleton of an object made of surfels*,
- a parallel algorithm for a *symmetric skeleton, which contains the topological axis* of an object made of surfels.

As far as we know, all the above algorithms have no equivalent.

Perspectives

- parallel 3D thinning
- parallel Euclidean skeletons
- general skeletons (*i.e.*, which are not necessarily principal subcomplexes)

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Questions