A Proof of Property 4

Proof. To prove Property 4, we prove its forward implication (Lemma 8) and its backward implication (Lemma 9).

Lemma 8. *Let f be a map from E into $\mathbb{R}^+$. If f is the saliency map of a hierarchical watershed of (G, w), then there exists an extinction map P of w such that, for any u in E, we have*

$$f(u) = \min\{P(R) \text{ such that } R \text{ is a child of } R_u\}.$$

Proof. Let us assume that the map f is the saliency map of a hierarchical $\mathcal{H}$ watershed of (G, w). Then, there exists a sequence of minima $\mathcal{S}$ such that $\mathcal{H}$ is the hierarchical watershed of (G, w) for $\mathcal{S}$. As stated in [7], the saliency map of $\mathcal{H}$ can be found trough the notion of persistence values *of the edges in E. Given any edge u in E, the persistence value of u (for $\mathcal{S}$) is the minimum extinction value among the children of R_u. Since the value of u in the saliency map f is precisely the persistence value of u, as established in [7], we have $f(u) = \min\{P(R)$ such that R is a child of $R_u\}$ where P is the extinction map of w for $\mathcal{S}$.* □

Lemma 9. *Let f be a map from E into $\mathbb{R}^+$. If there exists an extinction map P of w such that, for any u in E, we have*

$$f(u) = \min\{P(R) \text{ such that } R \text{ is a child of } R_u\},$$

then the map f is the saliency map of a hierarchical watershed of (G, w).

Proof. Let P be an extinction map of w such that, for any u in E, we have $f(u) = \min\{P(R)$ such that R is a child of $R_u\}$. If P is an extinction map of w, then there exists a sequence $\mathcal{S}$ of minima of w such that P is the extinction map of w for $\mathcal{S}$. Therefore, for any edge u in E, the value $f(u)$ is the persistence value of u for $\mathcal{S}$. As established in [7], the map f is the saliency map of the hierarchical watershed of (G, w) for $\mathcal{S}$. □

B Proof of Lemma 6

In order to prove Lemma 6, we first establish the following property which characterizes extinction maps.

Property 10. *Let P be a map from $\mathscr{R}(\mathcal{B})$ to $\mathbb{R}^+$. The map P is an* extinction map for *w if the following statements hold true:*

1. *$range(P) = \{0, \ldots, n\}$;*
2. *for any two minima M_1 and M_2 if $P(M_1) = P(M_2)$, then $M_1 = M_2$; and*
3. *for any region R of $\mathcal{B}$, we have $P(R) = \vee\{P(M)$ such that M is a minimum of w included in $R\}$.*

Proof. *Let P be a map from $\mathscr{R}(\mathcal{B})$ to $\mathbb{R}^+$ for which the statements 1, 2 and 3 hold true. To prove that P is an extinction map, we have to show that there exists a sequence S of n pairwise distinct minima of w such that, for any region R of $\mathcal{B}$, the value $P(R)$ is the extinction value of R for $\mathcal{S}$.*

Let $\mathcal{S} = (M_1, \ldots, M_n)$ be a sequence of n pairwise distinct minima of w ordered in non-decreasing order for P, i.e., *for any two distinct minima M_i and M_j, for i and j in $\{1, \ldots, n\}$, if $i < j$ then $P(M_i) \leq P(M_j)$.*

By the statement 2, the sequence $\mathcal{S}$ is unique. By the statement 3, for any region R of $\mathcal{B}$ such that there is no minimum of w included in R, $P(R) = \vee\{\} = 0$, so $P(R)$ is the extinction value of R for $\mathcal{S}$.

Since w has n minima, for any minimum M of w, the value $P(M)$ is in $\{1, \ldots, n\}$. Otherwise, if there were a minimum M' of w such that $P(M') = 0$, then there would be a value i in $\{1, \ldots, n\}$ such that for any minimum M'' of w the value $P(M'')$ is different from i. Consequently, the range of P would be $\{0, \ldots, n\} \setminus \{i\}$, which contradicts the statement 1. Therefore, for any minimum M_i, for i in $\{1, \ldots, n\}$, we have that $P(M_i) = i$, so $P(M_i)$ is the extinction value of M_i for S.

It follows that, by the statement 3, for any region R such that there is a minimum of $\mathcal{B}$ included in R, the value $P(R)$ is the maximum value i in $\{1, \ldots, n\}$ such that M_i is included in R.

Thus, for any region R of $\mathcal{B}$, the value $P(R)$ is the extinction value of R for $\mathcal{S}$. Therefore, the map P is an extinction map of w. □

Let f be a one-side increasing map for $\mathcal{B}$. To prove that the estimated extinction map ξ_f of f is an extinction map of w (Lemma 6), we introduce Lemmas 11, 12 and 16 which establish that the three statements of Property 10 hold for ξ_f.

Important notation: in the sequel, for any region X in $\mathscr{R}^*(\mathcal{B})$, we denote by u_X the building edge of X. For any region Y such that $Y \subseteq X$, we say that *Y is a descendant of X.*

Lemma 11. *Let f be a one-side increasing map for $\mathcal{B}$. The range of the estimated extinction map ξ_f of f is $\{0, \ldots, n\}$.*

Proof. *We have to show that:*

1. *for any i in $\{0, \ldots, n\}$, there is a region X of $\mathcal{B}$ such that $\xi_f(X) = i$; and*
2. *for any region X of $\mathcal{B}$, we have $\xi_f(X)$ in $\{0, \ldots, n\}$.*

Proof of 1: *By Definition 5, we have $\xi_f(V) = n$ and $\xi_f(Y) = 0$ for any region Y in $\mathscr{R}(\mathcal{B}) \setminus \mathscr{R}^*(\mathcal{B})$. Since f is a one-side increasing map, we know that, for any edge u, the value $f(u)$ is nonzero if and only if u is a watershed edge of w. Moreover, we know that the range of f is $\{0, \ldots, n-1\}$. Therefore, we can conclude that the $n-1$ watershed edges of w have pairwise distinct values in f ranging from 1 to $n-1$. Thus, for any i in $\{1, \ldots, n-1\}$, there is a region R_u such that u is in $WS(\mathcal{B})$ and such that $f(u) = i$. Since u is in $WS(\mathcal{B})$, none* *of the children of R_u is a leaf region. Therefore, there is a child Y of R_u such that $\xi_f(Y) = f(u)$ (Third case of Definition 5).*

Proof of 2: *By Definition 5, we know that the value $\xi_f(X)$ for any region X of $\mathcal{B}$ is either n (first case), 0 (second case), or $f(u)$ (third case), where u is the building edge of the parent of X. Therefore, the range of ξ_f is the union of range of f with $\{0, n\}$, which is precisely the set $\{0, \ldots, n\}$.* □

Lemma 12. *Let f be a one-side increasing map for $\mathcal{B}$ and let ξ_f be the estimated extinction map for f. For any two minima M_1 and M_2 of w, if $\xi_f(M_1) = \xi_f(M_2)$, then $M_1 = M_2$.*

To prove Lemma 12, we first present the Properties 13, 14 and 15.

Property 13. *Let f be a one-side increasing map and let ξ_f be the estimated extinction map of f. For any region X such that there is a minimum of w strictly included in X, there is a child Y of X such that:*

- $\xi_f(Y) = f(u_X)$;
- $\xi_f(sibling(Y)) = \xi_f(X)$; *and*
- *there is a minimum of w included in $sibling(Y)$.*

Proof. Let X be a region such that there is at least one minimum of w strictly included in X. Given a child Y of X:

- *If Y is a leaf region of $\mathcal{B}$, then there is no minimum of w included in Y and $\xi_f(Y) = 0$ (second case of Definition 5). It follows that u_X is not a watershed edge of w and that $f(u_X) = 0 = \xi_f(Y)$. Moreover, if Y is a leaf region of $\mathcal{B}$, then $\xi_f(sibling(Y)) = \xi_f(Y)$ (fourth case of Definition 5). Since there is no minimum of w included in Y, there is at least one minimum of w included in $sibling(Y)$.*
- *If Y is not a leaf region but $sibling(Y)$ is a leaf region of $\mathcal{B}$, then this is equivalent to the previous case. Otherwise, let us consider that Y and $sibling(Y)$ are not leaf regions of $\mathcal{B}$. This implies that there are minima of w included in both Y and $sibling(Y)$. By contradiction, let us assume that $\xi_f(Y) = \xi_f(sibling(Y)) = f(u_X)$. This implies that:*
  - **(a)** *$\vee\{f(v)$ such that R_v is a descendant of $Y\} < \vee\{f(v)$ such that R_v is a descendant of $sibling(Y)\}$* **or (b)** *$\vee\{f(v)$ such that R_v is a descendant of $Y\} = \vee\{f(v)$ such that R_v is a descendant of $sibling(Y)\}$ and $Y \prec sibling(Y)$;* **and**
  - **(c)** *$\vee\{f(v)$ such that R_v is a descendant of $sibling(Y)\} < \vee\{f(v)$ such that R_v is a descendant of $Y\}$* **or (d)** *$\vee\{f(v)$ such that R_v is a descendant of $sibling(Y)\} = \vee\{f(v)$ such that R_v is a descendant of $Y\}$ and $sibling(Y) \prec Y$.*

  However, the assertions **(a) and (c)**, **(a) and (d)**, **(b) and (c)**, *and* **(b) and (d)** *are contradictory. Therefore, we have either $\xi_f(Y) = \xi_f(X)$ or $\xi_f(sibling(Y)) = \xi_f(X)$. We can use a similar argument to prove that we have either $\xi_f(Y) = f(u_X)$ or $\xi_f(sibling(Y)) = f(u_X)$. Therefore, we may conclude that there is a child Y of X such that $\xi_f(Y) = f(u_X)$ and $\xi_f(sibling(Y)) = \xi_f(X)$.* □

Property 14. *Let u be any watershed edge of w and let f be a one-side increasing map. There is a minimum M of w such that $\xi_f(M) = f(u)$.*

Proof. Let u be a watershed edge of w and let f be a one-side increasing map. By Property 13, there is a child X_1 of R_u such that $\xi_f(X_1) = f(u)$. Since u is a watershed edge, X_1 cannot be a leaf node. If X_1 is a minimum of w, then the property holds true. Otherwise, by Property 13, there is a child X_2 of X_1 such that $\xi_f(X_2) = \xi_f(X_1) = f(u)$ and such that there is a minimum of w included in X_2. We can see that we define a sequence $(X_1, \ldots, X_p)$ where X_p is a minimum of w and such that $\xi_f(X_p) = \cdots = \xi_f(X_1) = f(u)$ and $X_i \subset X_{i-1}$ for any i in $\{2, \ldots, p\}$. Therefore, there is a minimum X_p included in R_u such that $\xi_f(X_p) = f(u)$. □

Property 15. *Let X be a region in $\mathscr{R}^*(\mathcal{B})$. There exists a minimum M of w such that $\xi_f(M) = \xi_f(X)$.*

Proof. If X is a minimum of w, then this is trivial. Otherwise, there is a minimum of w strictly contained in X. By Property 13, there is a child X_1 of X such that $\xi_f(X_1) = \xi_f(X)$ and such that there is a minimum of w included in X_1. If X_1 is a minimum of w, then the property holds true. Otherwise, by Property 13, there is a child X_2 of X_1 such that $\xi_f(X_2) = \xi_f(X_1) = \xi_f(X)$ and such that there is a minimum of w included in X_2. We can see that we define a sequence $(X_1, \ldots, X_p)$ where X_p is a minimum of w and such that $\xi_f(X_p) = \cdots = \xi_f(X_1) = \xi_f(X)$ and $X_i \subset X_{i-1}$ for any i in $\{2, \ldots, p\}$. Therefore, there is a minimum X_p included in X such that $\xi_f(X_p) = \xi_f(X)$. □

Proof (of Lemma 12).

Let f be a one-side increasing map for $\mathcal{B}$ and let ξ_f be the estimated extinction map for f. We need to prove that, for any two minima M_1 and M_2 of w, if $\xi_f(M_1) = \xi_f(M_2)$, then $M_1 = M_2$. By Property 14, we know that for any wateshed edge u of w, there is a minimum M such that $\xi_f(M) = f(u)$. By Property 15, we can say that there is a minimum M of w such that $\xi_f(M) = \xi_f(V) = n$. Since the range of f for the set of watershed edges is $\{1, \ldots, n-1\}$, we can conclude, by Properties 14 and 15, that the range of ξ_f for the set of minima of w is $\{1, \ldots, n\}$. Since w has n minima, it implies that the values $\xi_f(M_1)$ and $\xi_f(M_2)$ should be distinct for any pair (M_1, M_2) of distinct minima of w. □

Lemma 16. *Let f be a one-side increasing map for $\mathcal{B}$ and let ξ_f be the estimated extinction map for f. For any region R in $\mathscr{R}(\mathcal{B})$, we have $\xi_f(R) = \vee\{\xi_f(M)$ such that M is a minimum of w included in $R\}$.*

To prove this lemma, we introduce properties 17 and 18.

Property 17. *Let f be a one-side increasing map and let X be a region of $\mathcal{B}$. Then $\xi_f(X) \geq \vee\{f(v) \mid R_v \subseteq X\}$.*

Proof. Let X be a region of $\mathcal{B}$. We will prove that this property holds in the four cases of Definition 5.

1. *If* $X = V$*, then* $\xi_f(X) = n$ *(first case of Definition 5). Since the range of* f *is* $\{0, \dots, n-1\}$*, we have* $\xi_f(X) \geq \vee\{f(v) \mid R_v \subseteq X\}$*.*
2. *If there is no minimum of* w *included in* X*, then* X *is a leaf region. Therefore* $\xi_f(X) = 0$ *(second case of Definition 5) and* $\{f(v) \mid R_v \subseteq X\} = \emptyset$*. Thus,* $\xi_f(X) \geq \vee\emptyset = 0$*.*
3. *If* $\xi_f(X) = f(parent(X))$*, then* $\vee\{f(v)$ *such that* R_v *is a descendant of* $X\} \leq \vee\{f(v)$ *such that* R_v *is a descendant of* $sibling(X)\}$ *(Third case of Definition 5). Since* f *is a one-side increasing map, then* $f(u_{parent(X)}) \geq \vee\{f(v)$ *such that* R_v *is included in* $Z\}$ *for a child* Z *of* $parent(X)$*. Consequently,* $f(u_{parent(X)}) \geq \vee\{f(v)$ *such that* R_v *is a descendant of* $X\}$ *and, therefore,* $\xi_f(X) = f(u_{parent(X)}) \geq \vee\{f(v)$ *such that* R_v *is a descendant of* $X\}$*.*
4. *If* $\xi_f(X) = \xi_f(parent(X))$*, then we will prove that* $\xi_f(X) \geq \vee\{f(v)$ *such that* R_v *is a descendant of* $X\}$ *by induction.*
   - *Base step: if* $parent(X)$ *is* V*, then* $\xi_f(X) = \xi_f(V) = n$ *and our property holds true.*
   - *Inductive step: if the property holds for* $parent(X)$*, then we have to show that it holds for* X *as well. If* $\xi_f(parent(X)) \geq \vee\{f(v)$ *such that* R_v *is a descendant of* $parent(X)\}$ *then* $\xi_f(X) = \xi_f(parent(X)) \geq \vee\{f(v)$ *such that* R_v *is a descendant of* $X\}$ *because every descendant of* X *is a descendant of* $parent(X)$ *as well.* □

Property 18. *Let* X *be a region in* $\mathscr{R}^*(\mathcal{B})$*. Then, for any region* Y *such that* $Y \subseteq X$*, the value* $\xi_f(Y)$ *is in* $\{\xi_f(X), 0\} \cup \{f(u) \mid R_u \subseteq X\}$

Proof. *By induction:*

- *Base step: if* X *is a minimum of* w*. Then, for any child* Y *of* X*,* $\xi_f(Y) = 0$ *which is in* $\{\xi_f(X), 0\} \cup \{f(u) \mid R_u \subseteq X\}$*.*
- *Inductive step: if* X *is not a minimum and the property holds for both children of* X*. By Property 13, we know that there is a child* Y *of* X *such that* $\xi_f(Y) = f(u_X)$ *and* $\xi_f(sibling(Y)) = \xi_f(X)$*. Therefore, for any region* Y *such that* $Y \subseteq X$*, the value* $\xi_f(Y)$ *is in* $\{\xi_f(Y), 0\} \cup \{f(u) \mid R_u \subseteq Y\} \cup \{\xi_f(sibling(Y)), 0\} \cup \{f(u) \mid R_u \subseteq sibling(Y)\} \cup \{\xi_f(X)\}$ *which is equivalent to* $\{\xi_f(X), 0\} \cup \{f(u) \mid R_u \subseteq X\}$*.* □

Proof (of Lemma 16). We can now prove that, for any region R *in* $\mathscr{R}(\mathcal{B})$*, we have* $\xi_f(R) = \vee\{\xi_f(M)$ *such that* M *is a minimum of* w *included in* $R\}$*. Given a region* X *of* $\mathcal{B}$*:*

- *If there is no minimum of* w *included in* X*, then* $\xi_f(X) = 0$ *(statement 2 of Property 10). Then,* $\xi_f(X) = \vee\{\xi_f(M)$ *such that* M *is a minimum of* w *included in* $R\} = \vee\emptyset = 0$
- *Otherwise, for any region* $Y \subseteq X$*,* $\xi_f(Y)$ *is in* $\{\xi_f(X), 0\} \cup \{f(u) \mid R_u \subseteq X\}$ *by Property 18. By Property 17,* $\xi_f(X) \geq \{f(v) \mid R_v \subseteq X\}$*. Therefore,* $\xi_f(X) \geq \xi_f(Y)$*. Then,* ξ_f *is increasing on the hierarchy* $\mathcal{B}$*,* i.e., *for any region* X*, we have* $\xi_f(X) = \vee\{\xi_f(Y) \mid Y \subseteq X\}$*. By Property 13, there is a minimum* M *of* w *such that* $\xi_f(X) = \xi_f(M)$*. Hence,* $\xi_f(X) = \vee\{\xi_f(Y) \mid Y \subseteq X$ *and* Y *is a minimum of* $w\}$*.* □

C Proof of Lemma 7

Proof. Let f be a one-side increasing map. We will prove that, for any u in E, we have

- $f(u) = min\{\xi_f(R)$ *such that* R *is a child of* $R_u\}$.

Let u be an edge in E. By Property 17, we can infer that $\xi_f(R_u) \geq f(u)$. By Property 13, we know that, for a child Y of R_u, we have $\xi_f(Y) = f(u)$ and $\xi_f(sibling(Y)) = \xi_f(R_u)$. Therefore, $min\{\xi_f(R)$ such that R is a child of $R_u\} = min\{\xi_f(Y), \xi_f(sibling(Y))\} = min\{f(u), \xi_f(R_u)\} = f(u)$ for a child Y of R_u. □

D Proof of Theorem 3

Proof. We prove the forward and backward implications of Theorem 3 in Lemma 19 and Lemma 23, respectively.

Lemma 19. *Let $\mathcal{H}$ be a hierarchy and let $\Phi(\mathcal{H})$ be the saliency map of $\mathcal{H}$. If the hierarchy $\mathcal{H}$ is a hierarchical watershed of (G, w), then $\Phi(\mathcal{H})$ is a one-side increasing map for $\mathcal{B}$.*

Let $\mathcal{H}$ be a hierarchical watershed of (G, w) and let $\Phi(\mathcal{H})$ be the saliency map of $\mathcal{H}$. To prove that $\Phi(\mathcal{H})$ is a one-side increasing map for $\mathcal{B}$, we prove in the following three properties that the statements of Definition 2 hold true for $\Phi(\mathcal{H})$.

Property 20. *Let $\mathcal{H}$ be a hierarchical watershed of (G, w) and let $\Phi(\mathcal{H})$ be the saliency map of $\mathcal{H}$. The range of $\Phi(\mathcal{H})$ is $\{0, \ldots, n-1\}$.*

Proof. We have to show that:

1. *for any edge u in E, we have $\Phi(\mathcal{H})(u)$ in $\{0, \ldots, n-1\}$; and*
2. *for any i in $\{0, \ldots, n-1\}$, there is an edge u in E such that $\Phi(\mathcal{H})(u) = i$.*

Let P be a extinction map of w such that $\Phi(\mathcal{H})(u) = \min\{P(R)$ such that R is a child of $R_u\}$ (by Property 4). Let $\mathcal{S} = (M_1, \ldots, M_n)$ be a sequence of pairwise distinct minima of w such that, for any region R of $\mathcal{B}$, the value $P(R)$ is the extinction value of R for $\mathcal{S}$.

Let u be an edge in E. If u is not a watershed edge, then there is at least one child X of R_u such that X is a leaf node of $\mathcal{B}$. Then, $P(X) = 0$ and $\Phi(\mathcal{H})(u) = min\{0, P(sibling(X))\} = 0$. Otherwise, if u is a watershed edge of w, then both children of R_u includes at least one minimum of w. Let Y and X be the children of R_u. Since the extinction values of the minima of w for $\mathcal{S}$ are in $\{1, \ldots, n\}$ and are pairwise distinct, the extinction values of Y and X are in $\{1, \ldots, n\}$ and are pairwise distinct as well because they include different minima of w. Then, the value $\Phi(\mathcal{H})(u) = min\{P(X), P(Y)\}$ is in $\{1, \ldots, n\}$. If $P(X) = n$ (resp. $P(Y) = n$), then $P(Y) \neq n$ (resp. $P(X) \neq n$), which implies that $\Phi(\mathcal{H})(u) = P(Y) \neq n$ (resp. $\Phi(\mathcal{H})(u) = P(X) \neq n$). Therefore, the range of $\Phi(\mathcal{H})$ for the set of watershed edges of w is $\{1, \ldots, n-1\}$. Then, we have a proof for the statement 1.

Now, we will prove the statement 2. For $i = 0$, there is at least one edge u in E which is not a watershed edge of w. As stated in the previous paragraph, $\Phi(\mathcal{H})(u)$ should be $i = 0$. For i in $\{1, \ldots, n-1\}$, we will start by showing that, for any pair $\{u, v\}$ of watershed edges, we have $\Phi(\mathcal{H})(u) \neq \Phi(\mathcal{H})(v)$. Let $\{u, v\}$ be a pair of watershed edges of w. If the intersection between R_u and R_v is empty, then there is no intersection between the sets of minima included in the children of R_u and R_v, which implies that the children of R_u and R_v have pairwise distinct extinction values and, then, we have $\Phi(\mathcal{H})(u) \neq \Phi(\mathcal{H})(v)$. Otherwise, if there is an intersection between R_u and R_v, this implies that either $R_u \subset R_v$ or $R_v \subset R_u$. Let us assume that $R_u \subset R_v$. Let X be the child of R_u such that $R_v \subseteq X$. If $P(X) > P(sibling(X))$, then $\Phi(\mathcal{H})(u) = P(sibling(X))$, which is different from $P(M)$ for any minimum M included in R_v. Otherwise, let us consider that $P(X) < P(sibling(X))$. Let Y be the child of R_v such that $\Phi(\mathcal{H})(v) = P(Y)$. Since $\Phi(\mathcal{H})(v) = P(Y)$, we know that $P(sibling(Y))$ is larger than $P(Y)$. Since $sibling(Y)$ is a subset of X as well, the extinction value of X is larger or equal to the extinction value of $sibling(Y)$ and, thus, the extinction value of X is also larger than the extinction value of Y. Therefore, $\Phi(\mathcal{H})(u) = P(X) > P(Y) = \Phi(\mathcal{H})(v)$. We may conclude that, for any pair $\{u, v\}$ of watershed edges, we have $\Phi(\mathcal{H})(u) \neq \Phi(\mathcal{H})(v)$. Since w has $n-1$ watershed edges, for any i in $\{1, \ldots, n-1\}$, there is an watershed edge u of w such that $\Phi(\mathcal{H})(u) = i$. □

Property 21. *Let $\mathcal{H}$ be a hierarchical watershed of (G, w) and let $\Phi(\mathcal{H})$ be the saliency map of $\mathcal{H}$. For any edge in E, $\Phi(\mathcal{H})(u) > 0$ if and only if u is in $WS(w)$.*

Proof. Let P be a extinction map of w such that $\Phi(\mathcal{H})(u) = \min\{P(R)$ *such that* R *is a child of* $R_u\}$ *(by Property 4).*

Let u be an edge in E. The edge u is not a watershed edge if and only if there is at least one child X of R_u such that X is a leaf node of $\mathcal{B}$. The extinction value of any leaf node of $\mathcal{B}$ is zero. So, the edge u is not a watershed edge if and only if there is at least one child X of R_u such that $P(X) = 0$ and, consequently, $\Phi(\mathcal{H})(u) = min\{0, P(sibling(X))\} = 0$.

Let u be an edge in E. The edge u is a watershed edge of w if and only if both children of R_u include at least one minimum of w. The extinction value of any region X which includes at least one minimum of w is in $\{1, \ldots, n\}$ because the extinction value of any minimum of w is in $\{1, \ldots, n\}$. Then, the edge u is a watershed edge of w if and only if the extinction values of both children of R_u are in $\{1, \ldots, n\}$, consequently, $\Phi(\mathcal{H})(u) > 0$. □

Property 22. *Let $\mathcal{H}$ be a hierarchical watershed of (G, w) and let $\Phi(\mathcal{H})$ be the saliency map of $\mathcal{H}$. For any u in E, there exists a child R of R_u such that $\Phi(\mathcal{H})(u) \geq \vee\{\Phi(\mathcal{H})(v)$ such that R_v is included in $R\}$.*

Proof. Let P be a extinction map of w such that $\Phi(\mathcal{H})(u) = \min\{P(R)$ *such that* R *is a child of* $R_u\}$ *(by Property 4). We may affirm that P is increasing on the hierarchy $\mathcal{B}$ in the sense that, given two regions X and Y of $\mathcal{B}$, if $Y \subseteq X$ then $P(Y) \leq P(X)$.*

Let u be en edge in E. If there is a child X of R_u such that X is a leaf node of $\mathcal{B}$, then $\Phi(\mathcal{H})(u) \geq \vee\{\Phi(\mathcal{H})(v)$ such that R_v is included in $X\} = \vee\{\} = 0$. Otherwise, let X be the child of R_u such that $\Phi(\mathcal{H})(u) = P(X)$. For any region Y such that $Y \subseteq X$, we have $P(Y) \leq P(X)$. Thus, for any edge v such that $R_v \subseteq X$, we have $\Phi(\mathcal{H})(v) \leq P(X)$. Therefore, there is a child X of R_u such that $\Phi(\mathcal{H})(u) \geq \vee\{\Phi(\mathcal{H})(v)$ such that R_v is included in $X\}$. □

Lemma 23. *Let $\mathcal{H}$ be a hierarchy and let $\Phi(\mathcal{H})$ be the saliency map of $\mathcal{H}$. If $\Phi(\mathcal{H})$ is a one-side increasing map for $\mathcal{B}$, then the hierarchy $\mathcal{H}$ is a hierarchical watershed of (G, w).*

Proof. *Let $\mathcal{H}$ be a hierarchy and let $\Phi(\mathcal{H})$ be the saliency map of $\mathcal{H}$. If $\Phi(\mathcal{H})$ is a one-side increasing map for $\mathcal{B}$, then $\Phi(\mathcal{H})(u) = min\{\xi_{\Phi(\mathcal{H})}(R)$ such that R is a child of $R_u\}$ by Lemma 7. By Lemma 6, the estimated extinction map $\xi_{\Phi(\mathcal{H})}$ of $\Phi(\mathcal{H})$ is an extinction map of w. Then, by the backward implication of Property 4, the saliency map $\Phi(\mathcal{H})$ is the saliency map of a hierarchical watershed of (G, w). Hence, the hierarchy $\mathcal{H}$ is a hierarchical watershed of (G, w).* □