# Discrete Surfaces and Frontier Orders

Xavier Daragon[(a,b)], Michel Couprie[(a,b)], and Gilles Bertrand[(a,b)]

(a) École Supérieure d'Ingénieurs en Électrotechnique et Électronique
Laboratoire $A^2SI$
2, boulevard Blaise Pascal, Cité DESCARTES, BP 99
93162 Noisy le Grand CEDEX, France
(b) IGM, Unité Mixte de Recherche CNRS-UMLV-ESIEE UMR 8049
{daragonx,coupriem,bertrand}@esiee.fr

**Abstract.** Many applications require the extraction of an object boundary from a discrete image. In most cases, the result of such a process is expected to be, topologically, a surface, and this property might be required in subsequent operations. However, only through careful design can such a guarantee be provided. In the present article we will focus on partially ordered sets and the notion of $n$-surfaces introduced by Evako *et al.* to deal with this issue. Partially ordered sets are topological spaces that can represent the topology of a wide range of discrete spaces, including abstract simplicial complexes and regular grids. It will be proved in this article that (in the framework of simplicial complexes) any $n$-surface is an $n$-pseudomanifold, and that any $n$-dimensional combinatorial manifold is an $n$-surface. Moreover, given a subset of an $n$-surface (an object), we show how to build a partially ordered set called frontier order, which represents the boundary of this object. Similarly to the continuous case, where the boundary of an $n$-manifold, if not empty, is an $(n-1)$-manifold, we prove that the frontier order associated to an object is a union of disjoint $(n-1)$-surfaces. Thanks to this property, we show how topologically consistent Marching Cubes-like algorithms can be designed using the framework of partially ordered sets.

## INTRODUCTION

Many image processing applications require the extraction of the boundary of an object from a digital image. In continuous spaces, the notion of topological $n$-manifold generalizes the notion of surface (which is a 2-dimensional object) to the $n$-dimensional case. A topological $n$-manifold (without boundary) is an object (*i.e.*, a set of points) such that a neighborhood of each point is homeomorphic to an open ball of $\mathbb{R}^n$. Furthermore, the boundary of any "regular" $n$-dimensional object is a topological $(n-1)$-manifold. One of the aims of this article is to prove that object boundaries can be extracted from an $n$-dimensional digital image which are guaranteed to be $(n-1)$-dimensional discrete surfaces.

Different notions of discrete surfaces have appeared in the last decades. Some of them are defined in the framework of **simplicial complexes** [1–3]. A simplicial complex consists of a finite number of simplexes which can be thought of as elementary building blocks, glued together to form the complex. In the framework of piecewise linear topology [1], the simplexes are points (0-simplexes), straight line segments (1-simplexes), triangles (2-simplexes), tetrahedra (3-simplexes), and so on. This geometrical interpretation allows to make connections between continuous and discrete notions. Nevertheless, a simplicial complex can also be thought of as just a finite set, the vertices, with certain specified subsets, the simplexes, an $n$-simplex being a simplex composed of $n$ vertices; this purely combinatorial approach can lead to simpler statements and proofs.

Let us here describe intuitively the notion of **combinatorial manifold**; a formal and truly combinatorial definition will be given in the body of the paper. In a simplicial complex, the *closed star* of a 0-simplex $S$, denoted by $\widehat{star}(S)$, is the set composed of all the simplexes which contain $S$, and of all the simplexes which are included in these simplexes (see Fig. 1.2). A simplicial complex is said to be a combinatorial $n$-manifold if the boundary of the closed star of any 0-simplex $S$ does not contain $S$ and if there exists a subdivision of this boundary and a subdivision of the boundary of an $n$-simplex which are piecewise linearly homeomorphic (see Fig. 1). It has been proved that combinatorial manifolds are equivalent to triangulated topological manifolds up to dimension 3 (see [4]). Nevertheless the use of subdivision for defining combinatorial manifolds complicates basic problems like testing whether a particular complex is a combinatorial manifold.

Simpler notions are often favored, in particular for algorithms and demonstrations relative to their validity, such as the notion of **pseudomanifold**. An $n$-pseudomanifold $P$ is, roughly speaking, a connected and homogeneously

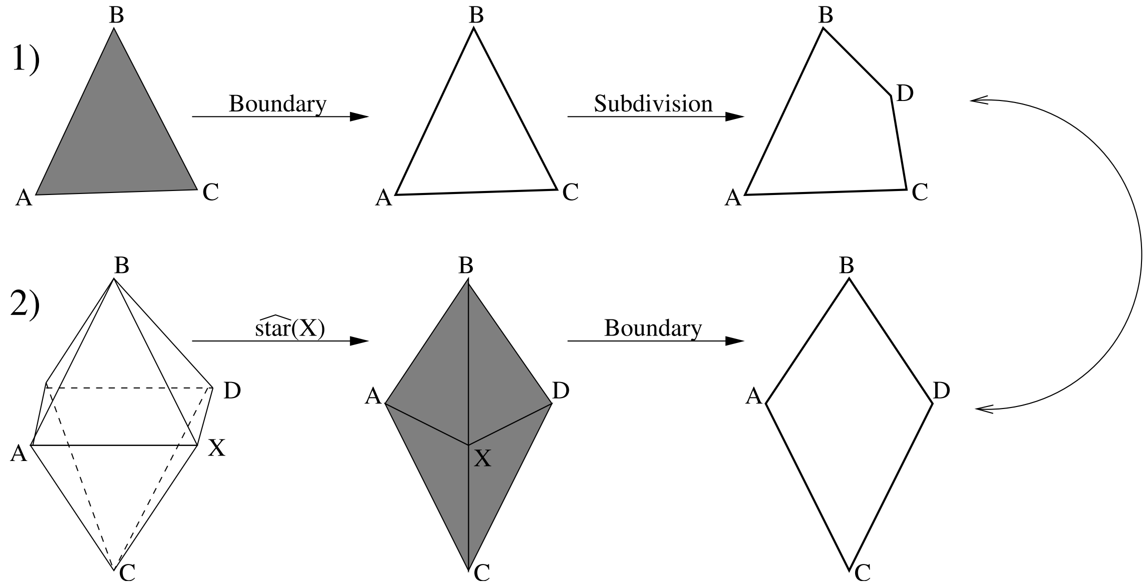



**Fig. 1.** Illustration of the notion of combinatorial manifold. 1) Any subdivision of the boundary of a 2-simplex is a 1-sphere. 2) In a 2-combinatorial manifold, the boundary of the closed star of each point is a 1-sphere.

$n$-dimensional simplicial complex. This means in particular that every simplex of $P$ is a subset of an $n$-simplex of $P$ and that every $(n-1)$-simplex of $P$ is a subset of exactly two $n$-simplexes of $P$. Pseudomanifolds of dimension 2 exist which are not combinatorial (nor topological) manifolds, like the pinched sphere depicted Fig. 2. The most important property related to pseudomanifolds is the Jordan-Brouwer theorem, which states that an $(n-1)$-pseudomanifold, embedded in the continuous space $\mathbb{R}^n$, separates $\mathbb{R}^n$ into two disjoint connected components. Most notions of discrete surfaces presented in the literature are deemed acceptable if they verify some analog of the Jordan-Brouwer theorem, and some authors prefer pseudomanifolds to combinatorial manifolds for they represent a wider class of "surfaces" verifying this theorem.

However, while combinatorial manifolds and pseudomanifolds are based upon simplicial complexes, most digital images are based upon a grid of square shaped cells. The framework of **digital topology** [5] has been developed to introduce topological notions (such as connectedness) in $\mathbb{Z}^n$ through adjacency graphs, and has led to many successful applications, mainly in 2D image analysis. The notion of 1-dimensional surface, or curve, is well defined in this framework and several notions of 2-dimensional surface have been developed [6–9]. Nevertheless, these notions cannot be extended to higher dimensions. In fact, the only attempt we know of at defining higher dimensional surfaces for digital topology [10] uses another framework, partially ordered sets. Notice that we only consider here approaches where the surface is defined as a subset of $\mathbb{Z}^n$; other approaches defining $n$-dimensional surfaces by the introduction of surface elements between the points of the space will be discussed later.

**Partially ordered sets** [11, 12, 10], also called orders or posets, are topological $T_0$ spaces [13]. A poset is a couple composed of a set and an order relation upon elements of this set. In such a space, two distinct elements are said to be neighbors if they are comparable. For example, the couple constituted by a simplicial complex and the inclusion relation is a poset. Khalimsky [11] has introduced a family of posets on $\mathbb{Z}^n$ which allows to formalize most of the topological notions useful in digital image analysis.

More recently, Evako has defined a simple and recursive notion of $n$-dimensional discrete surface in the framework of graphs, which may be extended to the framework of posets: **$n$-surfaces** [14–16]. A connected poset is an $n$-surface, for any $n > 0$, if the neighborhood of each element of this poset is an $(n-1)$-surface, a 0-surface being composed of two disconnected elements. It should be noted that Evako *et al.* [14] have proved that the space $\mathbb{Z}^n$ equipped with the Khalimsky topology is an $n$-surface, for any $n > 0$. Moreover, $n$-surfaces have been proved to verify analogs of the Jordan-Brouwer theorem in $\mathbb{Z}^2$ [17] and $\mathbb{Z}^3$ [18] equipped with the Khalimsky topology. Furthermore, the recursivity of the notion of $n$-surface is essential to demonstrate $n$-dimensional properties. This is the notion of surface upon which we will focus in this article.

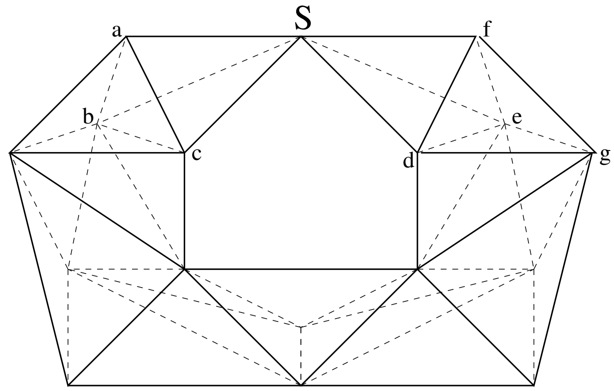



**Fig. 2.** This simplicial complex is a 2-dimensional pseudomanifold, called a pinched sphere.

Our interest in this paper is on the boundaries of discrete objects, which we want to define in such a way that we can prove that they are $n$-dimensional surfaces. The notion of boundary has been studied from different viewpoints.

The **internal boundary** [5] (see Fig. 3.2) and the **external boundary** (see Fig. 3.3), are among the earliest attempts to define discrete boundaries. A point of an object belongs to the internal boundary of this object if at least one of its neighbors belongs to the complementary of this object. The external boundary of the object is the internal boundary of its complementary. We can easily see that those definitions do not guarantee boundaries to be surfaces of any kind.

Another approach is to define the boundary as a set of surface elements separating the object from the rest of the image. In particular, **digital boundaries** [19–22] are constituted by pairs of points, representing the surface element between them, one of those points belonging to the object and the other to its complementary. It has been proved that digital boundaries are "near-Jordan" [19]: they separate $\mathbb{Z}^n$ in two disjoint domains (see Fig. 3.4 for an example). Stronger guarantees for these boundaries have been obtained but they require additional notions and hypotheses [22, 23].

The **Marching Cubes algorithm** [24] extracts a polygonal mesh representing the surface of an object in $\mathbb{Z}^3$ by generating a surface patch for each unit cube delimited by 8 points of $\mathbb{Z}^3$, depending on the status of these points. The original Marching Cubes algorithm is fast, but holes might appear in the generated surfaces. This problem was solved by, among others, J.-O. Lachaud [25], using the framework of digital topology, through the definition of continuous analogs of digital boundaries.

In previous articles [26, 27], we have introduced **frontier orders** (see Fig. 3.5) as a mean to extract the boundary of an object in any partially ordered set. As mentioned previously, partially ordered sets cover a wider range of spaces than digital topology.

The main contributions of this article are the following original results:

• We investigate the join operator (denoted by $*$) which is a fundamental tool to demonstrate properties relative to $n$-surfaces. Evako *et al.* [14] have proved that the join of any $n$-surface and any $m$-surface is an $(n+m+1)$-surface. We prove that, moreover, *$A * B$ is an $(n+1)$-surface if and only if $A$ is an $m$-surface and $B$ is an $(n-m)$-surface, for some $m \in \{0 \dots n\}$* (Th. 2), a property which is essential for the proof of following results.

• We prove, using the previous property, that *the simplicial complex defined by the fully ordered subsets of an order is an $n$-surface if and only if this order is itself an $n$-surface* (Th. 12), which will, in particular, result in Cor. 21 (see below).

• We establish, in the framework of abstract simplicial complexes, that *all $n$-combinatorial manifolds are $n$-surfaces* and that *all $n$-surfaces are $n$-pseudomanifolds* (Th. 17), implying that any $(n-1)$-surface, embedded in $\mathbb{R}^n$, verifies the Jordan-Brouwer theorem.

• We prove that *the frontier order of an object is a union of disjoint $(n-1)$-surfaces if the order to which the object belongs is an $n$-surface* (Th. 19 and Cor. 21).

• We describe how *a Marching Cubes-like algorithm coherent with the Khalimsky topology on $\mathbb{Z}^n$ can be implemented based on frontier orders.* The topology of the extracted surface is guaranteed by Th. 19.

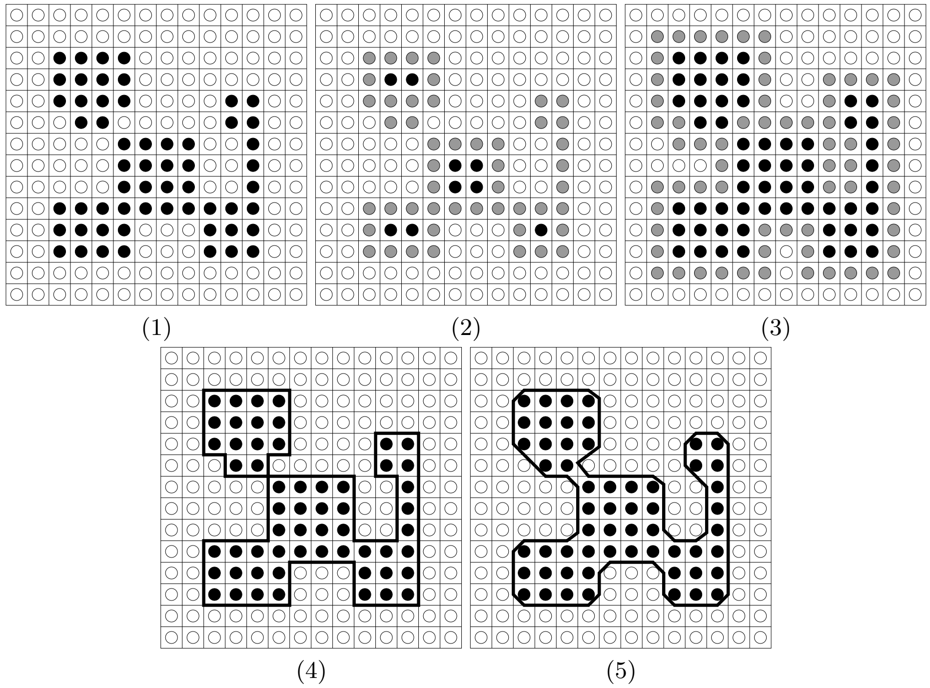



**Fig. 3.** Boundaries of an object. (1) Object $X$ (black dots). (2) Internal boundary of $X$ (grey dots). (3) External boundary of $X$ (grey dots). (4) Digital boundary of $X$ (thick line). (5) Frontier order of $X$ (thick line).

# 1 SURFACES AND PARTIALLY ORDERED SETS

This section introduces basic definitions and properties related to partially ordered sets and $n$-surfaces. Among these properties Th. 2 will play a central role in the demonstration of the main results of this paper.

## 1.1 Preliminary Definitions

Let us first introduce the notations we will use in this article. We write $S \subset X$ if $S$ is a subset of $X$ and $S \neq X$, in which case we say that $S$ is a *proper subset* of $X$. We write $S \subseteq X$ if $S \subset X$ or $S = X$. If $X$ is a set and $S \subseteq X$, when no confusion may occur we denote by $\overline{S}$ the complement of $S$ in $X$. If $\lambda$ is a binary relation on $X$, *i.e.*, a subset of the cartesian product $X \times X$, the *inverse* of $\lambda$ is the binary relation $\{(x,y) \in X \times X; (y,x) \in \lambda\}$. For any binary relation $\lambda$, $\lambda^\square$ is defined by $\lambda^\square = \lambda \setminus \{(x,x); x \in X\}$. For each $x$ of $X$, $\lambda(x)$ denotes the set $\{y \in X; (x,y) \in \lambda\}$ and for any subset $S$ of $X$, $\lambda(S)$ denotes the set $\{y \in \lambda(s); s \in S\}$.

An *order* is a pair $(X, \alpha_X)$ where $X$ is a set and $\alpha_X$ is a reflexive (for any $x$ in $X$, $x \in \alpha_X(x)$), antisymmetric (if $x \in \alpha_X(y)$ and $y \in \alpha_X(x)$ then $x = y$) and transitive (if $x \in \alpha_X(y)$ and $y \in \alpha_X(z)$ then $x \in \alpha_X(z)$) binary relation on $X$. When no confusion may occur, the order $(X, \alpha_X)$ will be simply denoted by $|X|$. Fig. 4.a depicts an example of order. Let $x$ be an element of $X$, the set $\alpha_X(x)$ is called the $\alpha_X$-*adherence* of $x$. We denote by $\beta_X$ the inverse of $\alpha_X$ and by $\theta_X$ the union of $\alpha_X$ and $\beta_X$. The set $\theta_X(x)$ is called the $\theta_X$-*neighborhood* of $x$, or simply the *neighborhood* of $x$ when no confusion may arise (see Fig. 4.b). We say that two elements $x$ and $y$ of $X$ are *neighbors*, or *comparable*, if $y \in \theta_X(x)$.
A *path* from $x_0$ to $x_n$ in $|X|$ is a sequence $(x_0, \ldots, x_n)$ of elements of $X$ such that $\forall i \in [1 \ldots n]$, $x_i \in \theta_X(x_{i-1})$; an example of path is depicted in Fig. 4.c. A *connected component* $C$ of $|X|$ is a maximal subset of $X$ such that for all $x, y \in C$, there exists a path from $x$ to $y$ in $C$.
Let $|X|$ be an order and let $x$ be an element of $X$. The *rank* of $x$ in $|X|$ is the number $\rho(x, |X|)$ such that $\rho(x, |X|) = 0$ if $\alpha_X^\square(x) = \emptyset$ and $\rho(x, |X|) = 1 + \max\{\rho(y, |X|), y \in \alpha_X^\square(x)\}$ otherwise. The *rank* of $|X|$ is the number $\rho(|X|)$ such that $\rho(|X|) = \max\{\rho(x, |X|), x \in X\}$.

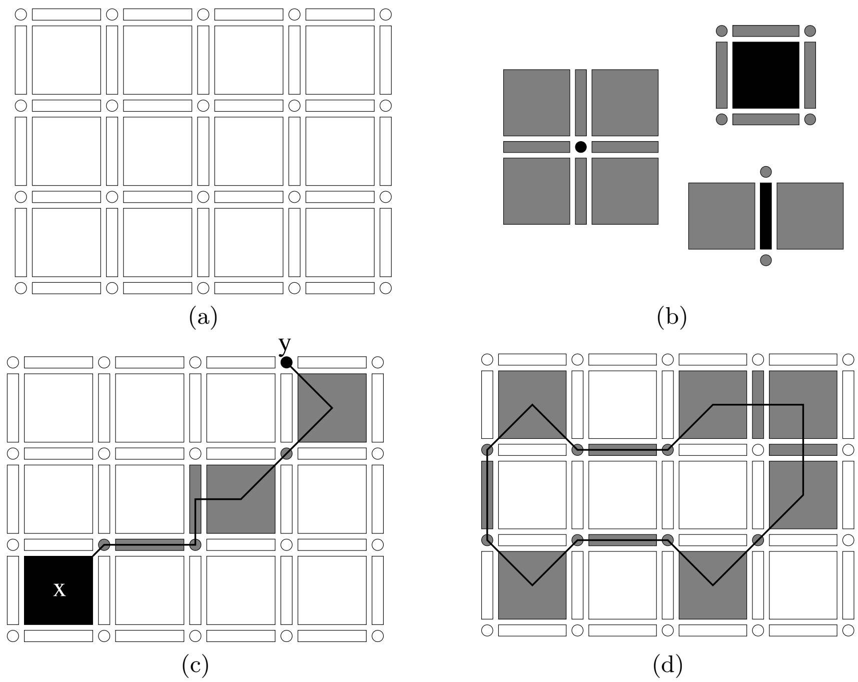



**Fig. 4.** a) An order $|X|$, which is composed of squares, their edges (depicted by thin rectangles) and their corners (depicted by small circles). If $x$ is a square of $|X|$, then $\alpha_X(x)$ is constituted by $x$ and the four edges and four corners which are adjacent to $x$. If $y$ is an edge in $|X|$, then $\alpha_X(y)$ is constituted by $y$ and the two corners adjacent to $y$. b) Three elements of $|X|$ (in black) and their $\theta$-neighborhoods (themselves and the grey elements surrounding them). c) A path from the black square $x$ to the black corner $y$. d) Example of 1-surface in the order $|X|$. This sub-order is of rank 1 even though it contains elements whose rank was 2 in the original order.

Any element of an order is called a *point* or an *n-element*, $n$ being the rank of this point. An element $x$ of an order $|X|$ is said to be *maximal* if $\beta^\Box_X(x) = \emptyset$; it is said to be *minimal* if $\alpha^\Box_X(x) = \emptyset$ (*i.e.*, $\rho(x, |X|) = 0$).
An order $|X|$ is *countable* if $X$ is countable, it is *locally finite* if, for each $x \in X$, $\theta_X(x)$ is a finite set. A *CF-order* is a countable locally finite order. In the sequel, we will consider only CF-orders.
Let $|X| = (X, \alpha_X)$ and $|Y| = (Y, \alpha_Y)$ be two orders, $|X|$ and $|Y|$ are *order isomorphic* if there exists a bijection $f : X \to Y$ such that, for all $x_1, x_2 \in X$, $x_1 \in \alpha_X(x_2) \Leftrightarrow f(x_1) \in \alpha_Y(f(x_2))$.
If $|X| = (X, \alpha_X)$ is an order and $S \subseteq X$, the *sub-order* of $|X|$ relative to $S$ is the order $(S, \alpha_S)$, with $\alpha_S = \alpha_X \cap (S \times S)$. When no confusion may arise, we also denote $|S| = (S, \alpha_S)$.

### 1.2 Discrete Surfaces

The notion of *n-surface*, whose definition in the framework of graphs has been proposed by Evako, Kopperman and Mukhin [14], will be one of the main focus of this article. Here, we restrict ourselves to the framework of orders, which can be seen as a particular class of graphs. As we will see in the sequel, important properties of $n$-surfaces can be easily proved in this purely discrete framework.

Let $|X| = (X, \alpha_X)$ be a non-empty CF-order.

• The order $|X|$ is a *0-surface* if X is composed of exactly two points $x$ and $y$ such that $y \notin \theta_X(x)$.

• The order $|X|$ is an *n-surface*, $n > 0$, if $|X|$ is connected and if, for each $x$ in $X$, the order $|\theta^\Box_X(x)|$ is an $(n-1)$-surface.

For technical reasons, we will say that the order $|X|$ is a $(-1)$-surface if $X = \emptyset$.

An example of 1-surface is given Fig. 4.d, in such a surface every point has exactly two $\theta^\Box$-neighbors. The order $|S2|$, depicted in Fig. 5.b, where $S2 = \{a, b, c, d, e, f\}$, $\alpha^\Box_{S2}(a) = \alpha^\Box_{S2}(b) = \emptyset$, $\alpha^\Box_{S2}(c) = \alpha^\Box_{S2}(d) = \{a, b\}$ and $\alpha^\Box_{S2}(e) = \alpha^\Box_{S2}(f) = \{a, b, c, d\}$, is the smallest 2-surface; it can be interpreted as a discrete representation of the tiling of a sphere depicted in Fig. 5.a. The Fig. 5.b shows a representation of this order as a directed graph, where

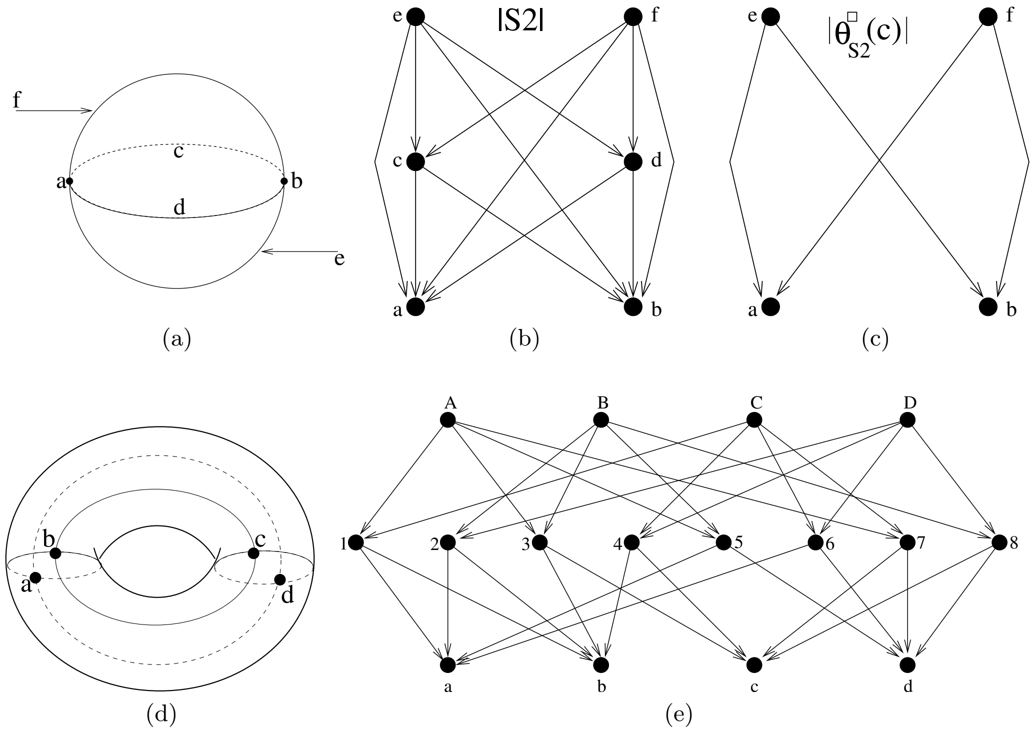



**Fig. 5.** a) Tiling of a (hollow) sphere constituted by two hemispheres $e$ and $f$ separated by a circle made from the segments $c$ and $d$ which are connected by the points $a$ and $b$. b) Graph representation of the order $|S2|$ which can be associated to the tiling (a). This is a 2-surface. c) $\theta^{\square}$-neighborhood of the 1-element $c$ in $|S2|$. This is a 1-surface. d) Tiling of a hollow torus. e) Graph representation of the order associated to the tiling (d). This is a 2-surface in which the 2-elements are denoted by $A$, $B$, $C$ and $D$, the 0-elements are denoted by $a$, $b$, $c$ and $d$, and the 1-elements are numbered from 1 to 8. Notice that, in order to ensure the readability of this figure, the edges which are induced by transitivity of the order relation are not depicted.

a directed edge $(x, y)$ is drawn between the elements $x$ and $y$ whenever $y \in \alpha^{\square}_{S2}(x)$. A slightly more complicated example of 2-surface is depicted in Fig. 5.e, which can be interpreted as the tiling of a torus depicted in Fig. 5.d.

We introduce now an important operator upon orders, which will be useful for demonstrating properties in any dimension.
Let $|X| = (X, \alpha_X)$ and $|Y| = (Y, \alpha_Y)$ be two orders such that $X \cap Y = \emptyset$. The *join* $|X| * |Y|$ is the order $(X \cup Y, \alpha_X \cup \alpha_Y \cup X \times Y)$. When no confusion may arise, we will sometimes write $X * Y$ instead of $|X| * |Y|$.
Observe that, if $X$ (*resp.* $Y$) is empty, then $|X| * |Y|$ equals $|Y|$ (*resp.* $|X|$).

Some basic examples of join can be found in Fig. 6, illustrating in particular the non-commutative nature of the join operator and the consequence of join upon the rank of elements. More precisely, it can be seen in Fig. 6.b and 6.c that, whenever joining an order $|L|$ to an order $|R|$, the rank of the elements of $|R|$ remains unchanged in $|L| * |R|$ while the rank of the elements of $|L|$ is incremented by $\rho(|R|) + 1$. The reader can also use Fig. 6 to find illustrations for the following property, which will be used to prove Th. 2.

**Property 1** *Let $|X|$ and $|Y|$ be two orders, with $X \cap Y = \emptyset$. Let $x$ be an element of $|X|$ and $y$ an element of $|Y|$. Then $|\theta^{\square}_{X*Y}(x)| = |\theta^{\square}_X(x)| * |Y|$ and $|\theta^{\square}_{X*Y}(y)| = |X| * |\theta^{\square}_Y(y)|$.*

**Proof:**
Notice that, if $x \in X$, any element of $Y$ belongs to $\theta_{X*Y}(x)$ and an element of $X$ belongs to $\theta^{\square}_{X*Y}(x)$ only if it belongs to $\theta^{\square}_X(x)$. Thus, using this fact and basic properties of set operators, we derive:

$$\begin{aligned}|\theta^{\square}_{X*Y}(x)| &= (\theta^{\square}_{X*Y}(x), (\alpha_X \cup \alpha_Y \cup X \times Y) \cap (\theta^{\square}_{X*Y}(x) \times \theta^{\square}_{X*Y}(x))) \\ &= (\theta^{\square}_{X*Y}(x), (\alpha_X \cap (\theta^{\square}_{X*Y}(x) \times \theta^{\square}_{X*Y}(x))) \cup (\alpha_Y \cap (\theta^{\square}_{X*Y}(x) \times \theta^{\square}_{X*Y}(x))) \\ &\quad \cup ((X \times Y) \cap (\theta^{\square}_{X*Y}(x) \times \theta^{\square}_{X*Y}(x))))\end{aligned}$$

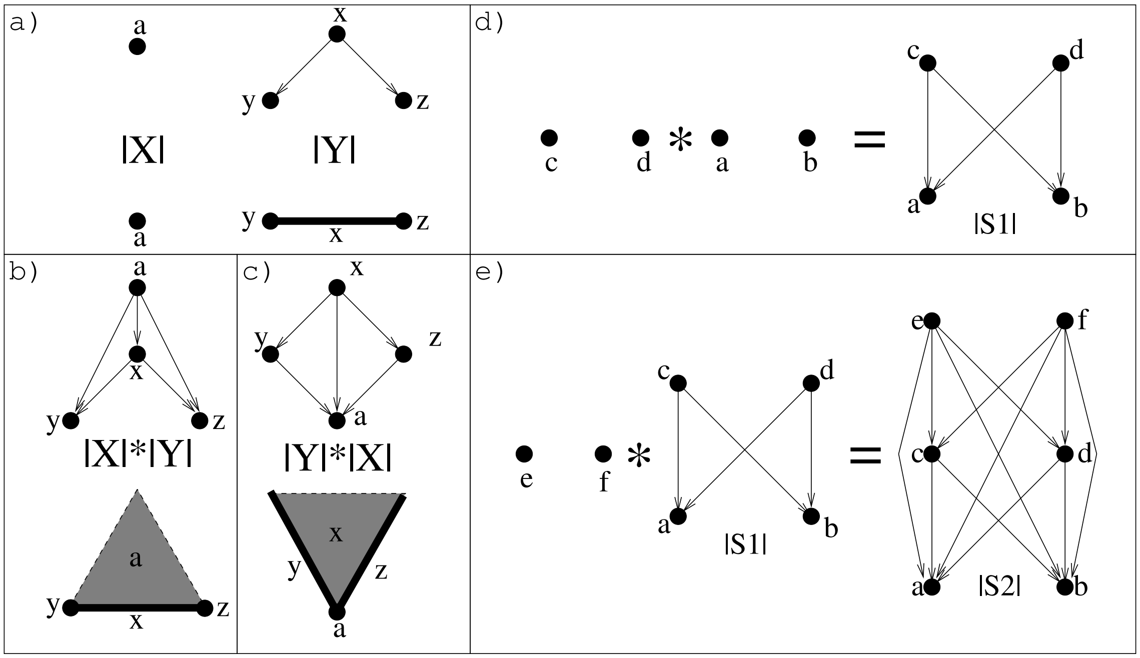



**Fig. 6.** a) The orders $|X| = (\{a\}, \emptyset)$ and $|Y| = (\{x, y, z\}, \{(x, y), (x, z)\})$ in graph (upper) and geometric (lower) representation.
b) The order $|X| * |Y|$ (graph and geometric representation).
c) The order $|Y| * |X|$ (graph and geometric representation).
d) Join between the 0-surfaces $(\{c, d\}, \emptyset)$ and $(\{a, b\}, \emptyset)$, resulting in the 1-surface $|S1| = (\{a, b, c, d\}, \{(c, a), (c, b), (d, a), (d, b)\})$.
e) Join between the 0-surfaces $(\{e, f\}, \emptyset)$ and the 1-surface $S1$. The result is the 2-surface $|S2|$ already depicted Fig. 5.b.

$$
\begin{aligned}
&= (\theta^{\square}_{X*Y}(x), (\alpha_{[X \cap \theta^{\square}_{X*Y}(x)]} \cup \alpha_{[Y \cap \theta^{\square}_{X*Y}(x)]} \cup (X \cap \theta^{\square}_{X*Y}(x)) \times (Y \cap \theta^{\square}_{X*Y}(x))) \\
&= (\theta^{\square}_{X}(x) \cup Y, \alpha_{\theta^{\square}_{X}(x)} \cup \alpha_Y \cup \theta^{\square}_{X}(x) \times Y) \\
&= |\theta^{\square}_{X}(x)| * |Y|
\end{aligned}
$$

The proof of $|\theta^{\square}_{X*Y}(y)| = |X| * |\theta^{\square}_{Y}(y)|$ is similar. $\square$

The following theorem turns the join operator into a major tool for demonstrating $n$-surfaces properties. It improves, in the framework of orders, a theorem proved by Evako *et al.* [14] in the framework of graphs, by introducing a necessary and sufficient condition. Examples of join upon $n$-surfaces can be found in Fig. 6.d and 6.e, where a 1-surface is obtained by joining two 0-surfaces together, and a 2-surface is obtained by joining a 0-surface with a 1-surface.

**Theorem 2** *Let $|X|$ and $|Y|$ be two orders, with $X \cap Y = \emptyset$ and let $n \in \mathbb{N}$. The order $|X| * |Y|$ is an $(n+1)$-surface if and only if $|X|$ and $|Y|$ are, respectively, $p$- and $(n-p)$-surfaces, with $-1 \leq p \leq n+1$.*

**Proof:**
The cases $p = -1$ (where $X = \emptyset$) and $p = n + 1$ (where $Y = \emptyset$) are immediate. So, from now on, we will suppose that neither $X$ nor $Y$ is the empty set.
It may be easily checked that the theorem is true for $n = 0$, suppose it is true for $n - 1$, with $n \geq 1$. Since neither $X$ nor $Y$ is the empty set, we observe that $|X| * |Y|$ is necessarily connected. Thus, from the very definition of an $(n+1)$-surface, we may affirm that $|X| * |Y|$ is an $(n+1)$-surface if and only if, for each $x \in X \cup Y$, $|\theta^{\square}_{X*Y}(x)|$ is an $n$-surface. By Prop. 1, and by the induction hypothesis, it means that $|X| * |Y|$ is an $(n+1)$-surface if and only if:
i) for each $x$ in $X$, $|\theta^{\square}_{X}(x)|$ and $|Y|$ are, respectively, $p'$- and $(n - 1 - p')$-surfaces; and
ii) for each $y$ in $Y$, $|X|$ and $|\theta^{\square}_{Y}(y)|$ are, respectively, $p''$- and $(n - 1 - p'')$-surfaces.
It may be seen that the condition "i) and ii)" is equivalent to "$|X|$ and $|Y|$ are, respectively, $p$- and $(n-p)$-surfaces", with $p = p'' = p' + 1$. $\square$

The following property, proved by Evako *et al.*, will be used in subsequent proofs.

**Property 3 (Evako *et al.* [14])** *The rank of any $n$-surface is precisely $n$.*

It may easily be checked that, for any element $x$ of an order $|X|$, we have $\theta^\square(x) = \beta^\square(x) * \alpha^\square(x)$. Thus, by Th. 2 and Prop. 3, we have:

**Property 4** *Let $|X| = (X, \alpha_X)$ be an order. Then, $|X|$ is an $n$-surface if and only if, for any $x$ in $X$, $|\alpha_X^\square(x)|$ is a $(k-1)$-surface and $|\beta_X^\square(x)|$ is an $(n-k-1)$-surface, with $k = \rho(x, |X|)$.*

The last properties of this section are discrete analogs of some basic manifold properties; their proofs can be found in the appendix at the end of this article.

Considering an element $x$ of an order $|X|$, it may be seen that $\rho(|\theta_X(x)|) = \rho(|\theta_X^\square(x)|) + 1$. Thus, a direct consequence of Prop. 3 is Cor. 5 which states that $n$-surfaces are "purely $n$-dimensional", in other words, each element of $x$ either has rank $n$ or has a neighbor of rank $n$.

**Corollary 5** *Let $|X|$ be an $n$-surface, each element $x$ of $X$ verifies $\rho(|\theta(x)|) = n$.*

Prop. 6 implies in particular that an $n$-surface cannot have a strict subset which is an $n$-surface, and thus is the only $n$-surface it contains.

**Property 6** *No proper sub-order of an $n$-surface is an $(n+k)$-surface, $n, k \geq 0$.*

Let $|X|$ be a connected order and let $Y \subset X$. We say that $|Y|$ *separates* $|X|$ if $|X \setminus Y|$ is not connected. Prop. 7 implies in particular that removing a point from a 2-surface or a curve from a 3-surface does not separate this surface into two disconnected components.

**Property 7** *Let $|X|$ be an $n$-surface, $n \geq 1$, and let $Y \subset X$. If $|Y|$ is a $k$-surface, $k \geq 0$, and if $|Y|$ separates $|X|$, then we have necessarily $k = n - 1$.*

## 2 SIMPLICIAL COMPLEXES, ORDERS AND SURFACES

Simplicial complexes are closely related to orders. Firstly, an (abstract) simplicial complex can be considered as a special case of order, hence, the notion of $n$-surface will be easily transposed to simplicial complexes. Secondly, to any order can be associated a simplicial complex called the chain complex of this order, and we will prove that, the chain complex associated to an order is an $n$-surface if and only if this order itself is an $n$-surface.
The results of this section will be used in the sequel of the paper, both to compare $n$-surfaces with combinatorial manifolds, and to establish the main properties of frontier orders.

### 2.1 Simplicial Complexes and Partially Ordered Sets

Let $\Lambda$ be a countable set, any non-empty subset of $\Lambda$ is called a *simplex*. A subset constituted of $(n+1)$ elements of $\Lambda$ is also called an *$n$-simplex*. Any non-empty subset of a simplex is called a *face* of this simplex. A *proper face* of a simplex is a non-empty proper subset of this simplex. Now, let $C$ be a family of simplexes of $\Lambda$, we say that $C$ is a *simplicial complex* if it is closed by inclusion, which means that, if $s$ belongs to $C$, then any face of $s$ also belongs to $C$. A *(simplicial) $n$-complex* is a simplicial complex in which maximal elements (for the inclusion) are $n$-simplexes. The minimal subset $\Lambda_C$ of $\Lambda$ such that any element of $C$ is a subset of $\Lambda_C$ is called the *support* of $C$. Let $C$ be a simplicial complex, any subset of $C$ which is also a simplicial complex is called a *subcomplex* of $C$.
The simplicial complexes we just defined are often called *abstract simplicial complexes*, as opposed to other notions of complexes based upon an underlying Euclidean space.

To any simplicial complex $C$, we can associate a canonical order $|C| = (C, \alpha_C)$ where $\alpha_C$ is the inclusion relation: $t \in \alpha_C(s)$ means that $t \subseteq s$. In this paper, we will often refer to the canonical order associated to a simplicial complex, especially when it allows simpler formulations or proofs. Let $C$ be a simplicial complex and let $s$ be a simplex of $C$. we observe that $\alpha_C(s)$ does not depend on $C$ since any simplicial complex is closed by inclusion. Thus we will often

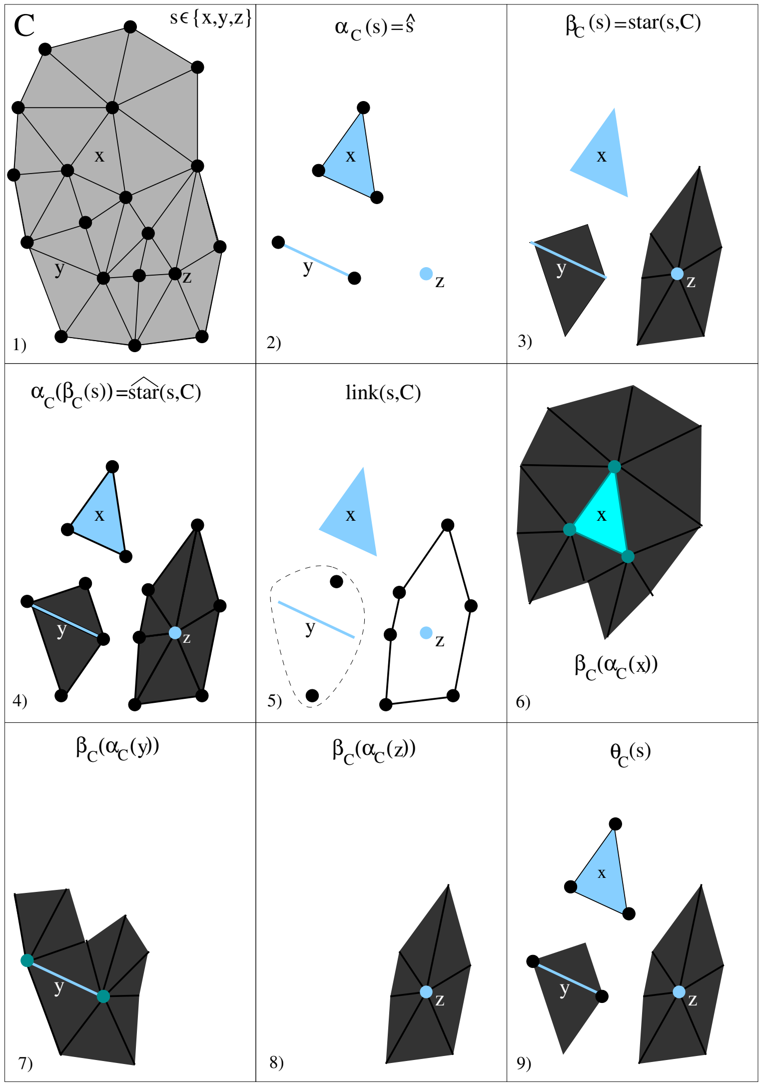



**Fig. 7.** 1) The simplicial complex $C$, in which $x$ is a 2-simplex, $y$ is a 1-simplex and $z$ is a 0-simplex.
2) Depicts $\widehat{x}$, $\widehat{y}$ and $\widehat{z}$, which are the same as $\alpha_C(x)$, $\alpha_C(y)$ and $\alpha_C(z)$.
3) Depicts $star(x, C)$, $star(y, C)$ and $star(z, C)$, which are the same as $\beta_C(x)$, $\beta_C(y)$ and $\beta_C(z)$.
4) Depicts $\widehat{star}(x, C)$, $\widehat{star}(y, C)$ and $\widehat{star}(z, C)$, which are the same as $\alpha_C(\beta_C(x))$, $\alpha_C(\beta_C(y))$ and $\alpha_C(\beta_C(z))$.
5) Depicts $x$ and $link(x, C)$ (which is empty), $y$ and $link(y, C)$ (two points) and $z$ and $link(z, C)$.
6) Depicts $\beta_C(\alpha_C(x))$. Associated to 4) we can see that $\alpha_C(\beta_C(x)) \setminus \beta_C(\alpha_C(x)) = link(x, C) = \emptyset$.
7) Depicts $\beta_C(\alpha_C(y))$. We can see that $\alpha_C(\beta_C(y)) \setminus \beta_C(\alpha_C(y)) = link(y, C)$.
8) Depicts $\beta_C(\alpha_C(z))$. We can see that $\alpha_C(\beta_C(z)) \setminus \beta_C(\alpha_C(z)) = link(z, C)$.
9) Depicts $\theta_C(x)$, $\theta_C(y)$ and $\theta_C(z)$.

write $\alpha$ instead of $\alpha_C$ when discussing about simplicial complexes. We say that the simplicial complex $C$ is *connected* if the order $|C|$ is connected, this is equivalent to the classical definition of connectedness of simplicial complexes. We can easily see that for any $n$-simplex of $C$, for any $n \geq 0$, we have $\rho(s, |C|) = n$.

Let us now introduce the main operators used in the framework of simplicial complexes, and explain how they are related to previously defined operators on partially ordered sets. These operators are illustrated in Fig. 7.

**Closure and Boundary**
The *closure* $\widehat{s}$ of the simplex $s$ is the simplicial complex consisting of $s$ and all its faces. By extension, the *closure* $\widehat{S}$

of a set of simplexes $S$ is the union of the closures of its simplexes. The *boundary* $\partial(s)$ of a simplex $s$ is the simplicial complex constituted by all the proper faces of $s$. In terms of order, we have $\widehat{s} = \alpha(s)$, $\widehat{S} = \alpha(S)$ and $\partial(s) = \alpha^{\square}(s)$

**Star**

The *(open) star* of the simplex $s$ in the simplicial complex $C$ is defined as $star(s, C) = \{t \in C; s \subseteq t\}$. Thus $star(s, C)$ is equal to $\beta_C(s)$. The *closed star* of $s$ in $C$ is defined as the closure of the star of $s$, that is, $\widehat{star}(s, C) = \{t \in C; \exists u \in star(s, C), t \subseteq u\}$. Notice that the star of a simplex is not a simplicial complex in general, while the closed star of a simplex is always a simplicial complex. In terms of order, we have $\widehat{star}(s, C) = \alpha_C(\beta_C(s))$.

**Simplicial join**

Two simplexes are *joinable* if their intersection is empty. If $s$ and $t$ are joinable simplexes, the *simplicial join* of $s$ and $t$ is defined as $s \circ t = s \cup t$. The simplicial join of a simplex $s$ with the empty set is defined as being $s$ itself: $s \circ \emptyset = s$.
Let $C$ and $K$ be two simplicial complexes, they are said to be *joinable* if every simplex of $C$ is joinable with every simplex of $K$. If $C$ and $K$ are joinable, the *simplicial join* of $C$ and $K$ is then defined as the simplicial complex $C \circ K = C \cup K \cup \{s \circ t, s \in C, t \in K\}$. Notice that, unlike the order join, the simplicial join is commutative.
While the simplicial join cannot be easily expressed in terms of order relation, Prop. 13, in section 2.3, will provide some insight into the relations between simplicial join and order join.

**Link**

The *link* of the simplex $s$ in the simplicial complex $C$ is defined as $link(s, C) = \{t \in C; s \circ t \in C\}$. Notice that this is always a simplicial complex.
A direct expression of the link operator in terms of order relation is provided by Prop. 8. However, it is usually more convenient to use the isomorphism introduced in Prop. 9; this isomorphism is illustrated in Fig. 8 (see appendix for proofs of both properties).

**Property 8** *Let $s$ be a simplex of a simplicial complex $C$, $link(s, C) = \alpha_C(\beta_C(s)) \setminus \beta_C(\alpha_C(s))$.*

**Property 9** *Let $s$ be a simplex of a simplicial complex $C$, $link(s, C)$ is (order) isomorphic to $\beta^{\square}_C(s)$.*

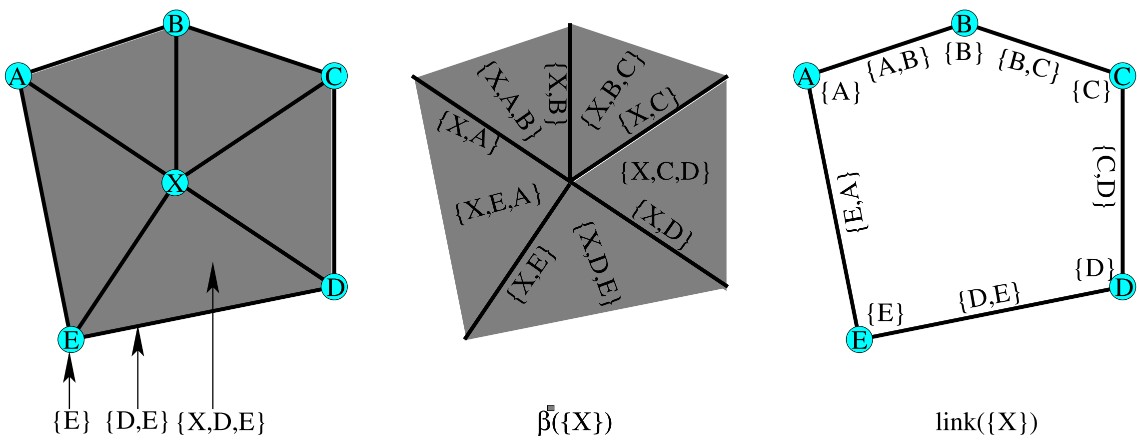



**Fig. 8.** The leftmost figure depicts the closed star of the 0-simplex $\{X\}$, which can be divided into $\{X\}$, $\beta^{\square}(\{X\})$ (center) and the link of $\{X\}$ (right). Notice that the link of $\{X\}$ can be obtained from $\beta^{\square}(\{X\})$ by factorization and that the link of $\{X\}$ and $\beta^{\square}(\{X\})$ are order-isomorphic.

## 2.2 Simplicial Complexes and $n$-Surfaces

A simplicial complex $C$ is said to be an *$n$-surface* if the order $(C, \subseteq)$ is an $n$-surface. We will see that it is possible to give a characterization of these particular $n$-surfaces, which is not based on order notions. Besides, some proofs related to $n$-surfaces are simpler in this framework.

Those familiar with the notion of (combinatorial) manifold, which will be discussed later in this article, know that the boundary of an $n$-simplex is used for deriving a general definition of combinatorial spheres. The next property establishes that this boundary is always an $(n-1)$-surface. While not surprising, this result is quite important since many proofs, in the framework of simplicial complexes, depend on this property (proof: see appendix).

**Property 10** *Let $s$ be an $n$-simplex, with $n \geq 0$, then $\alpha^{\Box}(s)$ (i.e., the boundary of $s$) is an $(n-1)$-surface.*

As shown by the following property (whose proof can be found in the appendix), a simpler characterization of $n$-surfaces can be derived from Prop. 10 in the case of simplicial complexes:

**Property 11** *Let $C$ be a connected simplicial complex of support $\Lambda$. The simplicial complex $C$ is an $n$-surface, with $n > 0$, if, and only if, $\forall x \in \Lambda$, $link(\{x\}, C)$ is an $(n-1)$-surface.*

### 2.3 Chains of an Order

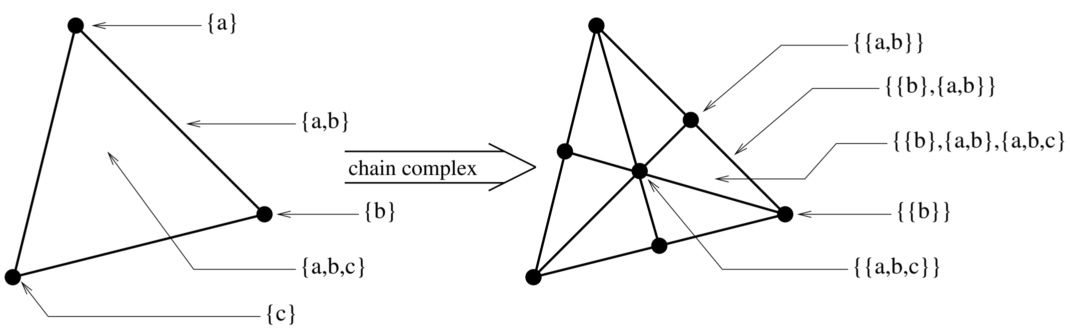



**Fig. 9.** Graphical illustration of the notion of chain complex. Left: the initial order $|X|$ which is constituted by the closure of the simplex $\{a, b, c\}$. Right: the chain complex $\mathcal{C}^X$ constituted by the chains of $|X|$.

Let $|X|$ be an order, a *chain* of $|X|$ is a fully ordered subset of $X$ (*i.e.*, a subset $S$ of $X$ such that any two elements of $S$ are comparable). An $n$-chain is a chain having $n+1$ elements. We call *chain complex of* $X$, and denote by $\mathcal{C}^X$, the set of all the chains of $|X|$, *i.e.*, $\mathcal{C}^X = \{S \subseteq X, \forall x_1, x_2 \in S, x_1 \in \theta_X(x_2)\}$. It should be noted that, for any order $|X|$, $(\mathcal{C}^X, \subseteq)$ is an order and that $\mathcal{C}^X$ is a simplicial complex, the support of which is $X$, as illustrated in Fig. 9. Moreover, the structure of $(\mathcal{C}^X, \subseteq)$ is strongly related to the structure of $|X|$, as shown by the following theorem:

**Theorem 12** *Let $|X|$ be an order. Then, the simplicial complex $\mathcal{C}^X$ formed by its chains is an $n$-surface if and only if $|X|$ is an $n$-surface.*

**Proof:**
This property is obvious for $n = 0$. Moreover the equivalence between $|X|$ connectedness and $\mathcal{C}^X$ connectedness is straightforward.
Let us now assume that the property is true for $n \geq 0$. By Prop. 11 we may affirm that $\mathcal{C}^X$ is an $(n+1)$-surface if, and only if, for any 0-simplex $s$ of $\mathcal{C}^X$, $link(s, \mathcal{C}^X)$ is an $n$-surface.
Let $s = \{x_0\}$ be a 0-simplex of $\mathcal{C}^X$, we have:

$$\begin{aligned} link(s, \mathcal{C}^X) &= \{\{x_1, \ldots, x_k\}, \{x_0, x_1, \ldots, x_k\} \in C^X\} \\ &= \mathcal{C}^{\theta_X^{\Box}(x_0)} \end{aligned}$$

Thus, we deduce from the induction hypothesis that $\mathcal{C}^X$ is an $(n+1)$-surface if, and only if, $|X|$ is an $(n+1)$-surface. $\Box$

While order relations do not allow a simple expression of the simplicial join operator, the next property shows a deep connection between the order join and the simplicial join (proof: see appendix).

**Property 13** *Let $|X|$ and $|Y|$ be two orders such that $X \cap Y = \emptyset$, then $\mathcal{C}^{X*Y} = \mathcal{C}^{Y*X} = \mathcal{C}^X \circ \mathcal{C}^Y$.*

## 3 DISCRETE SURFACES, MANIFOLDS AND PSEUDOMANIFOLDS

The notion of $n$-surface was introduced quite recently and comparisons with more classical notions of discrete surfaces are lacking. In this section, we provide an analysis of the relationships between combinatorial manifolds, pseudomanifolds and $n$-surfaces; and obtain a classification theorem.

### 3.1 Pseudomanifolds

A simplicial $n$-complex $C$ is said to be *pure* if each of its simplexes is a face of an $n$-simplex of $C$. A pure $n$-complex $C$ is said to be *strongly connected* if every two $n$-simplexes of $C$ can be connected by a sequence of $n$-simplexes such that any two consecutive simplexes have an $(n-1)$-dimensional face in common. More precisely, in terms of order, a pure $n$-complex $C$ is strongly connected if for any two distinct $n$-elements of the order $|C| = (C, \subseteq)$, there exists a path $(x = x_0, \ldots, x_{2r} = y)$ in $|C|$ with $r \in \mathbb{N}$ such that $\forall i \in [0, r]$, $x_{2i}$ is an $n$-element and $\forall j \in [0, r[$, $x_{2j+1}$ is an $(n-1)$-element (notice that this definition also makes sense for orders which are $n$-surfaces but not simplicial complexes). A strongly connected $n$-dimensional complex $C$ is called an *$n$-pseudomanifold (without boundary)* if every $(n-1)$-simplex of $C$ is a face of exactly two $n$-simplexes.

### 3.2 $n$-Surfaces and Pseudomanifolds

As shown by Cor. 5, every simplicial complex which is an $n$-surface is pure. We will now show that simplicial complexes which are $n$-surfaces have strong connectedness and pseudomanifold properties.

**Property 14** *Let $C$ be a simplicial complex. If $C$ is an $n$-surface, $n > 0$, then $C$ is strongly connected.*

**Proof:**
We will prove that this property is true for all $n$-surfaces (whether they are simplicial complexes or not). Let $x$ and $y$ be two distinct $n$-elements of $C$. Since $x$ and $y$ both have rank $n$, we know that $y \notin \theta_C(x)$.
If $|C|$ is a 1-surface, then the proof of this property is straightforward. Let us now assume that the property is true for any $k$, $0 < k < n$ and let $|C|$ be an $n$-surface:
• Since $|C|$ is an $n$-surface, $n > 0$, $|C|$ is connected and there exists a path $\pi^0 = (x = x_0^0, \ldots, x_r^0 = y)$ from $x$ to $y$ in $|C|$ (Fig. 10.1).
• By transitivity of the order relation we can suppress all intermediary elements from $\pi^0$ and we obtain a path $\pi^1 = (x = x_0^1, \ldots, x_{2m}^1 = y)$ such that $\forall i \in ]0, m]$, $x_{2i-1}^1 \in \alpha_C^\square(x_{2i}^1)$ and $\forall i \in [0, m[$, $x_{2i+1}^1 \in \alpha_C^\square(x_{2i}^1)$ (Fig. 10.2).
• By transitivity, and thanks to the purity of $n$-surfaces (Cor. 5), we can now obtain a path $\pi^2 = (x = x_0^2, \ldots, x_{2m}^2 = y)$ such that $\forall i \in ]0, m]$, $x_{2i-1}^2 = x_{2i-1}^1$ and $\forall i \in ]0, m[$, $x_{2i}^2$ is an $n$-element of $|C|$ in $\beta_C^\square(x_{2i}^1)$ (Fig. 10.3).
• Now, let $i \in ]0, m]$, and let us consider the element $x_{2i-1}^2$ of $\pi^2$. Since $|C|$ is an $n$-surface, $|\theta_C^\square(x_{2i-1}^2)|$ is an $(n-1)$-surface. The rank of $x_{2i-1}^2$ is either equal to $n-1$ or strictly lower than $n-1$. In the latter case, by induction hypothesis, there exists a path $\pi' = (x_{2i-2}^2 = z_0, \ldots, z_{2p} = x_{2i}^2)$ from $x_{2i-2}^2$ to $x_{2i}^2$ in $|\theta_C^\square(x_{2i-1}^2)|$ such that $\forall j \in [0, p]$, $z_{2j}$ is an $(n-1)$-element of $|\theta_C^\square(x_{2i-1}^2)|$ (and thus an $n$-element of $|C|$), and $\forall j \in [0, p[$, $z_{2j+1}$ is an $(n-2)$-element of $|\theta_C^\square(x_{2i-1}^2)|$ (and thus an $(n-1)$-element of $|C|$).
• By replacing every element of rank strictly lower than $n-1$ in $\pi^2$ by a path $\pi'$ built as described previously, we obtain a path $\pi^3 = (x = x_0^3, \ldots, x_{2q}^3 = y)$, $q \in \mathbb{N}$, such that $\forall i \in [0, q]$, $x_{2i}^3$ is an element of rank $n$ and $\forall i \in [0, q[$, $x_{2i+1}^3$ is an element of rank $(n-1)$ (Fig. 10.4). □

It is now easy to prove that any $n$-surface is an $n$-pseudomanifold: Prop. 4 implies that, for any $(n-1)$-simplex $s$ in a simplicial complex $C$ which is an $n$-surface, $\beta_C^\square(s)$ is a 0-surface. Thus, every $(n-1)$-simplex of $C$ is a face of exactly two $n$-simplexes of $C$ and, as a consequence of Prop. 14 and Prop. 4, we have:

**Property 15** *Let $C$ be a simplicial complex. If $C$ is an $n$-surface, $n > 0$, then $C$ is an $n$-pseudomanifold without boundary.*

The converse of Prop. 15 is not true. A counter-example is given by the pinched sphere (Fig. 2), which is a 2-pseudomanifold without boundary but not a 2-surface, for the neighborhood of the point $S$ is not a 1-surface (it is not connected).

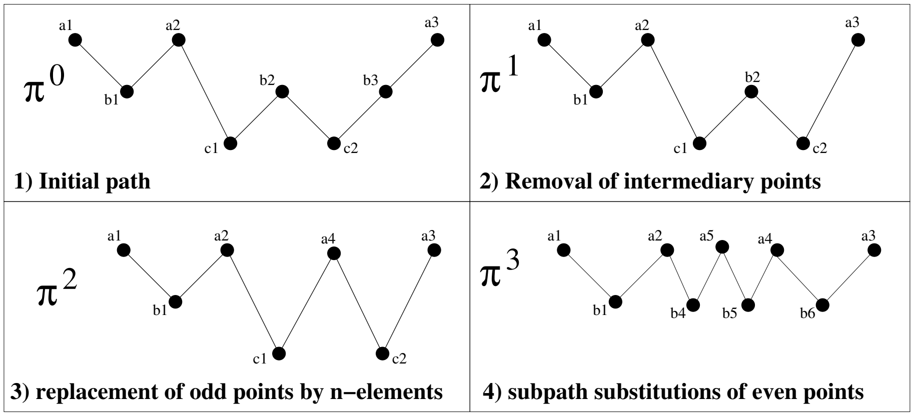



**Fig. 10.** Illustration for the proof of Prop. 14.

### 3.3 Manifolds

In the introduction, we gave an informal presentation of combinatorial manifolds. We will here provide the necessary definitions related to stellar manifolds, which have been proved (in [3]) to be equivalent to combinatorial manifolds [1]. For the sake of simplicity we will use the word manifold as a shortcut for stellar manifold in the sequel.

Let $s$ be a simplex in a simplicial complex $C$ and $\{x\}$ a 0-simplex not in $C$. The *stellar subdivision* of $s$ (in $C$) at point $\{x\}$ is defined by $(s,\{x\})_C = (C \setminus \widehat{star}(s,C)) \cup (\{x\} \circ \alpha^\square_C(s) \circ link(s,C))$. This operation is illustrated Fig. 11. As can be seen in the second example, the stellar subdivision of a given simplex may affect its surrounding, more exactly its star. The inverse operation, denoted by $(s,\{x\})^{-1}_C$ is called a *stellar weld*. More precisely, if $\{x\} \in C$ and $s \notin C$ is such that there exists a subcomplex $t \in C$ for which $link(\{x\},C) = \alpha^\square(s) \circ t$, $(s,\{x\})^{-1}_C = (C \setminus \widehat{star}(\{x\},C)) \cup (s \circ t)$. If the simplicial complex $C$ can be obtained from the simplicial complex $D$ by a sequence of stellar subdivisions and welds, $C$ and $D$ are said to be *stellar equivalent*. In the following definitions, $s^n$ will denote an $n$-simplex. An *$n$-sphere* is a simplicial complex which is stellar equivalent to the boundary of $s^{n+1}$ (*i.e.*, to $\alpha^\square(s^{n+1})$). An *$n$-manifold (without boundary)* $M$ is a connected simplicial complex such that for every 0-simplex $\{v\}$ of $M$, $link(\{v\},M)$ is a stellar $(n-1)$-sphere.

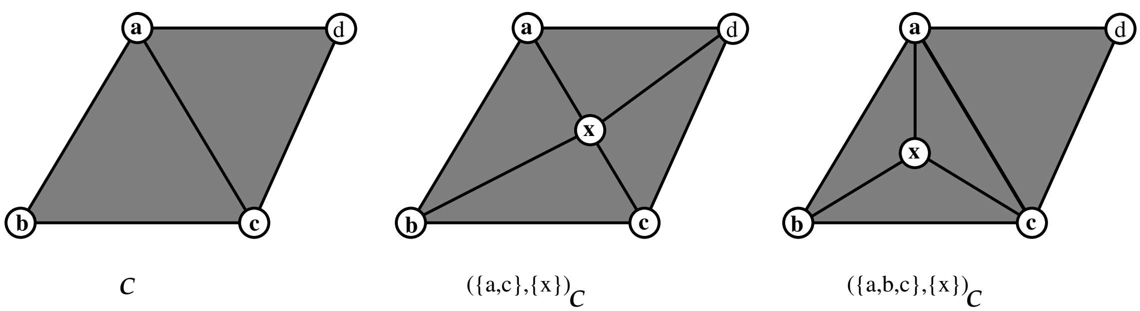



**Fig. 11.** Examples of stellar moves. Left: original simplicial complex $C$. Center: result of the stellar subdivision $(\{a,c\},\{x\})_C$. Right: result of the stellar subdivision $(\{a,b,c\},\{x\})_C$.

### 3.4 $n$-Surfaces and Manifolds

**Property 16** *Let $M$ be an $n$-manifold without boundary, with $n > 0$, then $M$ is an $n$-surface.*

**Proof:**
The following proof will use the property (which can be found in [1]) that an $n$-sphere, with $n > 0$, is an $n$-manifold without boundary.
A 0-surface and a 0-sphere are both made of two isolated points.
Let $M$ be a 1-manifold, by definition the link of every 0-simplex of $M$ is a 0-sphere. Thus it is a 0-surface and, according to Prop. 11, $M$ is a 1-surface.
Let us now assume that the property is true for any $n$-manifold and let $M$ be an $(n+1)$-manifold. By definition, the link of any 0-simplex $x$ of $M$ is an $n$-sphere. Then, since an $n$-sphere is an $n$-manifold, it is an $n$-surface by induction hypothesis. Thus, according to Prop. 11, $M$ is an $(n+1)$-surface. □

As it was the case with Prop. 15, the converse of Prop. 16 is not true. A counter-example is the order join of a 0-surface and the torus depicted in Fig. 5.d, which is a 2-surface. The result is a 3-surface, according to Th. 2, and so is its chain complex (Th. 12). Yet, this chain complex is not a 3-manifold, since (roughly speaking) there are two points whose neighborhood is a torus and not a sphere.
Nevertheless, since 0-spheres are equivalent to 0-surfaces, it can be deduced that 1-manifolds without boundary are equivalent to 1-surfaces. Furthermore, it can easily be seen that all 1-manifolds without boundary are 1-spheres, thus all 2-surfaces are 2-manifolds without boundary.

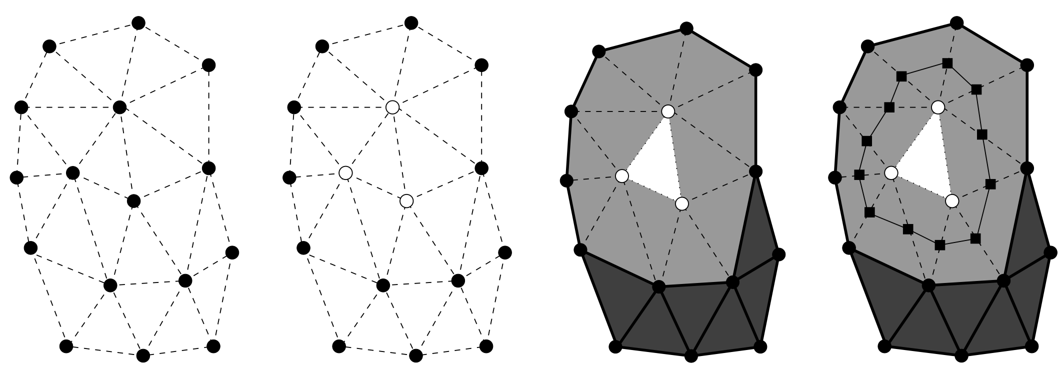
### 3.5 Concluding Property

The following theorem summarizes this entire section, establishing a classification between pseudomanifolds, manifolds and $n$-surfaces.

**Theorem 17** *Let $n \in \mathbb{N}^*$, let $P_n$ be the set of all $n$-pseudomanifolds without boundary, $S_n$ the set of all the simplicial complexes which are $n$-surfaces and $M_n$ the set of all $n$-manifolds without boundary. We have $M_1 = S_1 = P_1$, $M_2 = S_2 \subset P_2$ and, for any $n > 2$, $M_n \subset S_n \subset P_n$.*

## 4 FRONTIER ORDERS

In former articles [26, 27], we have introduced frontier orders as a way of extracting boundaries of objects in arbitrary orders. The aim of this section is to prove that, in any $n$-surface $|X|$, the frontier order associated to any subset of $X$ is a union of disjoint $(n-1)$-surfaces.



a) b) c) d)

**Fig. 12.** a) A simplicial complex $C$ with support $X$. b) The point set $X$ is partitioned into two subsets: $K$ [object, white points] and $\overline{K}$ [background, black points]. c) This bi-partition of the point set induces a tri-partition of the simplicial complex between an object complex [white], a background complex [black] and a frontier order (this is not a simplicial complex) [grey]. d) Embedding of the frontier order [black squares and the lines connecting them].

### 4.1 Definition

If we consider a simplicial complex $C$ with support $X$ (Fig. 12.a), the partition of $X$ between a set $K$, *the object*, and its complementary $\overline{K}$, *the background* (Fig. 12.b), induces a partition of $C$ into three sets (Fig. 12.c):

- $C_K$, the set of all the simplexes which are subsets of $K$
- $C_{\overline{K}}$, the set of all the simplexes which are subsets of $\overline{K}$
- $C_{K/\overline{K}}$, the set of the simplexes being neither subset of $K$ nor subset of $\overline{K}$

We call *frontier order* of $K$ in $X$, relatively to $C$, the suborder $|C_{K/\overline{K}}|$ of $|C|$ relative to $C_{K/\overline{K}}$. Notice that, since a singleton (0-simplex) is either a subset of $K$ or a subset of $\overline{K}$, $C_{K/\overline{K}}$ is not closed for the inclusion and, consequently, is not a simplicial complex.

### 4.2 Surface Properties

By definition, $|C_{K/\overline{K}}|$ is both *symmetrical*, since $C_{K/\overline{K}} = C_{\overline{K}/K}$, and *separating*, since any path from any $x \in K$ to any $y \in \overline{K}$ crosses $C_{K/\overline{K}}$. The purpose of this section is to prove that, if a simplicial complex is an $n$-surface, $n > 1$, then the frontier order induced by any bi-partition of its support is a union of disjoint $(n-1)$-surfaces.
This proof will be divided, so that it might be more easily understood. First, we will prove that the frontier order associated to the bi-partition of the vertices of the boundary of an $n$-simplex is an $(n-2)$-surface.

**Property 18** *Let $s$ be an $n$-simplex, $n \geq 2$, and let the simplicial complex $C = \alpha^{\Box}(s)$ be the boundary of $s$. Let $K$ be a non-empty proper subset of $s$. Then the frontier order $|C_{K/\overline{K}}|$ is an $(n-2)$-surface.*

**Proof:**
We first examine the case $n = 2$. In this case, $s$ is of the form $\{x, y, z\}$. Up to permutation and complementation we can assume that $K = \{x, y\}$, in which case $|C_{K/\overline{K}}| = (\{\{x, z\}, \{y, z\}\}, \subseteq)$, a 0-surface.
Now, we assume that the property is true for all $i \leq n$, with $n \geq 2$. Let $s$ be an $(n+1)$-simplex and let $K$ be a non-empty proper subset of $s$. Let $t$ be a $k$-simplex, $k \leq n$, $t \in C_{K/\overline{K}}$ (notice that $k \geq 1$).

**(a)** By Prop. 10, $|C| = |\alpha^{\Box}(s)|$ is an $n$-surface.
**(b)** By definition of surfaces, we know that $|\theta_C^{\Box}(t)|$ is an $(n-1)$-surface.
**(c)** By Prop. 10, $|\alpha_C^{\Box}(t)|$ is a $(k-1)$-surface.
**(d)** By induction hypothesis at rank $k$, $|[\alpha_C^{\Box}(t)]_{K/\overline{K}}|$ is a $(k-2)$-surface.
**(e)** By Th. 2, $|\beta_C^{\Box}(t)|$ is an $(n-k-1)$-surface *(from (b) and (c))*.
**(f)** Moreover, since $t \in C_{K/\overline{K}}$, $|\beta_C^{\Box}(t)| = |[\beta_C^{\Box}(t)]_{K/\overline{K}}|$.
**(g)** So, $|[\theta_C^{\Box}(t)]_{K/\overline{K}}| = |[\beta_C^{\Box}(t)]_{K/\overline{K}}| * |[\alpha_C^{\Box}(t)]_{K/\overline{K}}|$ is an $(n-2)$-surface *(from (e), (f) and (d))*.

Consequently $|C_{K/\overline{K}}|$ is a union of disjoint $(n-1)$-surfaces.
We now have to prove that $|C_{K/\overline{K}}|$ is connected, *i.e.*, that any two elements of $|C_{K/\overline{K}}|$ are linked by a path in $|C_{K/\overline{K}}|$. Let us consider two simplexes $s_1$ and $s_2$ of $|C_{K/\overline{K}}|$. There exists four points $a$, $b$, $c$ and $d$ (not necessarily all distinct) such that $a \in s_1 \cap K$, $b \in s_1 \cap \overline{K}$, $c \in s_2 \cap K$ and $d \in s_2 \cap \overline{K}$. Suppose that $\{a, b, c, d\} = s$, then it may be verified that $|C_{K/\overline{K}}| = (\{\{a, b\}, \{a, b, c\}, \{b, c\}, \{b, c, d\}, \{c, d\}, \{c, d, a\}, \{d, a\}, \{d, a, b\}\}, \subseteq)$, which is obviously a connected 1-surface. Otherwise, $\{a, b, c, d\}$ is a proper subset of $s$ and belongs to $C_{K/\overline{K}}$, and $(s_1, \{a, b\}, \{a, b, c, d\}, \{c, d\}, s_2)$ is a path from $s_1$ to $s_2$ in $|C_{K/\overline{K}}|$. $\Box$

The previous property being established for the boundary of an $n$-simplex, we can now prove the main theorem of this section:

**Theorem 19** *Let $C$ be a simplicial complex which is an $n$-surface, $n > 1$, and let $X$ be its support. Let $K$ be a non-empty proper subset of $X$. Then the frontier order $|C_{K/\overline{K}}|$ is a union of disjoint $(n-1)$-surfaces.*

**Proof:**
Let $S$ be a $k$-simplex of $C_{K/\overline{K}}$ (thus $k \geq 1$). In order to establish the theorem, it is sufficient to prove that for any $S$ in $C_{K/\overline{K}}$, $|\theta_{C_{K/\overline{K}}}^{\Box}(S)|$ is a $(n-2)$-surface. We observe that:

**(a)** By definition of surfaces, we know that $|\theta_C^{\Box}(S)|$ is an $(n-1)$-surface.
**(b)** By Prop. 10, $|\alpha_C^{\Box}(S)|$ is a $(k-1)$-surface.

**(c)** By Prop. 18, $|[\alpha^\square_C(S)]_{K/\overline{K}}|$ is a $(k-2)$-surface.

**(d)** By Th. 2, $|\beta^\square(S)|$ is an $(n-k-1)$-surface *(from (a) and (b))*.

**(e)** Moreover, since $S \in C_{K/\overline{K}}$, $|\beta^\square_C(S)| = |[\beta^\square_C(S)]_{K/\overline{K}}|$.

**(f)** Now, using Th. 2, we deduce that $|[\theta^\square_C(S)]_{K/\overline{K}}| = |[\beta^\square_C(S)]_{K/\overline{K}}| * |[\alpha^\square_C(S)]_{K/\overline{K}}|$ is an $(n-2)$-surface *(from (d), (e) and (c))*.

Moreover, $[\theta^\square_C(S)]_{K/\overline{K}} = \{e \in \theta^\square_C(S), e \not\subseteq K, e \not\subseteq \overline{K}\}$ and $\theta^\square_{C_{K/\overline{K}}}(S) = \{e \in \theta^\square_C(S)\} \cap \{e \in C, e \not\subseteq K, e \not\subseteq \overline{K}\}$, so $\theta^\square_{C_{K/\overline{K}}}(S) = [\theta^\square_C(S)]_{K/\overline{K}}$. We conclude that for any $S$ in $C_{K/\overline{K}}$, $|C|$ being an $n$-surface, $|\theta^\square_{C_{K/\overline{K}}}(S)|$ is an $(n-2)$-surface. $\square$

We give now a result which relates the connectedness of the object to the connectedness of its boundary. Of course, a connected object may have a disconnected boundary (think about a ball with a cavity), but we will show that, roughly speaking, a disconnected object must have a disconnected boundary.

Let $C$ be a simplicial complex and let $X$ be its support. Let $K$ and $L$ be two disjoint subsets of $X$, we say that $K$ and $L$ are *unrelated* (in $C$) if $\forall s \in K$, $\forall t \in L$, $\{s,t\}$ is not a simplex of $C$.

Let $|X| = (X, \alpha_X)$ be an order. We say that two suborders $|K|$ and $|L|$ of $|X|$ are *adjacent* if, for some $x \in K$ and some $y \in L$, we have $x \in \theta_X(y)$. We say that two suborders $|K|$ and $|L|$ of $|X|$ are *mutually disconnected* if they are not adjacent.

**Property 20** *Let $C$ be a simplicial complex and let $X$ be its support. Let $K$ and $L$ be two unrelated subsets of $X$. Then $C_{K/\overline{K}}$ and $C_{L/\overline{L}}$ are mutually disconnected.*

### 4.3 Partially Ordered Sets and Frontier Orders

Using chain complexes, frontier orders can be defined for any partially ordered set. More exactly, if $|X|$ is a partially ordered set and $K$ is a subset of $X$ we define the frontier order of $K$ in $|X|$ as $|C^X_{K/\overline{K}}|$ where $C^X$ is the chain complex of $|X|$. The following property is a direct consequence of Th. 12 and Th. 19:

**Corollary 21** *Let $|X| = (X, \alpha_X)$ be an order and $K$ a non-empty proper subset of $X$. If $|X|$ is an $n$-surface then the frontier order $|C^X_{K/\overline{K}}|$ is a union of disjoint $(n-1)$-surfaces.*

Prop. 22 and Cor. 23 relate the connectedness of a suborder to the connectedness of its chain complex and its frontier order, respectively. The proofs are straightforward and will be ommited.

**Property 22** *Let $|X|$ an order, let $|K|$ and $|L|$ be two suborders of $|X|$ (hence $K$ and $L$ are subsets of the support of $C^X$, the chain complex of $|X|$). Then $K$ and $L$ are unrelated in $C^X$ if and only if $|K|$ and $|L|$ are mutually disconnected.*

**Corollary 23** *Let $|X|$ be an order, let $|K|$ and $|L|$ be two mutually disconnected sub-orders of $|X|$. Then, the frontier orders $|C^X_{K/\overline{K}}|$ and $|C^X_{L/\overline{L}}|$ are mutually disconnected.*

## 5 MARCHING CUBES-LIKE ALGORITHM

The goal of this section is to give an application of the preceding theory to the fields of computer graphics and visualization of discrete objects. There is a number of theoretical, but also practical works (see *e.g.* [28–34]) which promote the frameworks of cell complexes, simplicial complexes and orders for image processing. Thus, the need for visualization programs for objects in such spaces is real.

Notice that an object in the three-dimensional Khalimsky grid $H^3$ may be obtained through different ways: in particular, by a transformation applied to a subset of $\mathbb{Z}^3$ (see [10] for examples of such transformations and connections with the "classical" digital topology), by discretization of an analytically specified geometrical form, using a specific discretization scheme (see [34]); or as the result of an operator acting in $H^3$ (*e.g.* skeletonization operators, see [30, 31]).

We provide here the first "marching cubes-like" algorithm for the three-dimensional Khalimsky grid, the proof of its correctness straightforwardly follows from Cor. 21.

The Marching Cubes algorithm [24] provides an efficient way to extract a polygonal surface from an object expressed as a subset of $\mathbb{Z}^3$, or an approximated isosurface from a regular (cubic grid based) sampling of a function

$\phi : \mathbb{R}^3 \to \mathbb{R}$. The main feature of this algorithm is a look-up table associating a surface patch to each possible partition of the corners of a unit cube between two sets of points. Given a map $f : \mathbb{Z}^3 \to \mathbb{R}$ and a value $n$, the Marching Cubes algorithm aims at building a frontier which separates the sets $K = \{x \in \mathbb{Z}^3, f(x) > n\}$ and $\overline{K} = \mathbb{Z}^3 \setminus K$. For each unit cube $c$ of the cubic grid $\mathbb{Z}^3$, the algorithm uses $c \cap K$ and $c \cap \overline{K}$ as a key for finding the appropriate surface patch in the look-up table, interpolates the position of its support points according to the values of the eight corners of this unit cube, and memorizes the resulting surface element. The union of all those elements constitutes a polygonal mesh, which is the result of the Marching Cubes algorithm. In the case of binary images, it is sufficient to take the look-up table elements as they are, without any interpolation.

Nevertheless, the polygonal meshes generated by this algorithm are not guaranteed to be sound surfaces. In particular, artefacts such as holes might appear. We will now detail how to design an $n$-dimensional Marching Cubes-like algorithm, based on frontier orders, in the case where the image is considered as a subset of $\mathbb{Z}^n$ equipped with the Khalimsky topology. The surfaces generated by our algorithm are guaranteed to be $(n-1)$-surfaces.

### 5.1 Embedding of the frontier order

Let us first notice that, given any set $X$ of points in $\mathbb{R}^n$ upon which a simplicial complex $C$ is built, the frontier order of a subset $K$ of $X$ can unambiguously be embedded in $\mathbb{R}^n$ using the following scheme:

- To any 0-element $\{p_0, p_1\}$ of $C_{K/\overline{K}}$ is associated the point $(p_0 + p_1)/2$.
- To any $n$-element of $C_{K/\overline{K}}$ is associated the convex hull of the points it contains.

Such an embedding is illustrated in Fig. 12.d.

Let us now introduce the Khalimsky grids as the family of orders $|H^n| = (H^n, \subseteq)$, defined by:

$$
\begin{aligned}
H^1_0 &= \{\{a\}, a \in \mathbb{Z}\} \\
H^1_1 &= \{\{a, a+1\}, a \in \mathbb{Z}\} \\
H^1 &= H^1_0 \cup H^1_1 \\
H^n &= \{h_1 \times \ldots \times h_n, \forall i \in [1, n], h_i \in H^1\}, n > 1
\end{aligned}
$$

In [14], Evako *et al.* have proved that $|H^n|$ is an $n$-surface for all $n \in \mathbb{N}^*$. By Cor. 21, this implies that the frontier defined for any subset of an order $|H^n|$ is a union of disjoint $(n-1)$-surfaces.

A natural encoding of the set $H^n$ by elements of $\mathbb{Z}^n$ is defined as follows [11]: to every element $h_1 \times \ldots \times h_n$ of $H^n$ is assigned the vertex of coordinates $(z_1, \ldots, z_n)$ in $\mathbb{Z}^n$, such that $\forall i \in [1 \ldots n]$, $z_i = 2v_i$ if $h_i = \{v_i\}$ and $z_i = 2v_i + 1$ if $h_i = \{v_i, v_i + 1\}$.

Then, as explained in section 4, we can use the chain complex of $|H^n|$ to define the frontier order of any subset $K$ of $H^n$, and we can use the embedding scheme introduced earlier:

- We have seen that 0-elements of a frontier order are of the form $\{A, B\}$. To each such element we assign the point (of $\mathbb{R}^n$) $(a + b)/2$ where $a = (a_1, \ldots, a_n)$ (*resp.* $b = (b_1, \ldots, b_n)$) is the vertex assigned to $A$ (resp. $B$) in $\mathbb{Z}^n$.
- To each 1-element we assign the segment joining the points associated to the 0-elements of its $\alpha$-adherence.
- To each 2-element we assign the corresponding polygon (which is in fact either a triangle or a parallelogram).
- And so on...

### 5.2 Look-up tables

In the 3-dimensional case, the embedding process described above results in the look-up table depicted in Fig. 13. It should be noted that, $|H^3|$ being an heterogeneous space, the corners of the unit cube are depicted using different shapes, so as to represent the different types of element composing $|H^3|$: the cube stands for a 3-element, the square faces stand for 2-elements, the cylinders stand for 1-elements and the sphere stands for a 0-element. Look-up tables corresponding to higher dimensions can easily be obtained in the same way, however the size of such tables grows exponentially as dimension increases.

Since they are based upon tetrahedra (3-chains) rather than directly upon cubes, the configurations shown in Fig. 13 are more facetized than those of the original Marching Cubes algorithm. It is however possible to simplify the surface patches associated to the various configurations, while ensuring that the overall topology of the surface is preserved

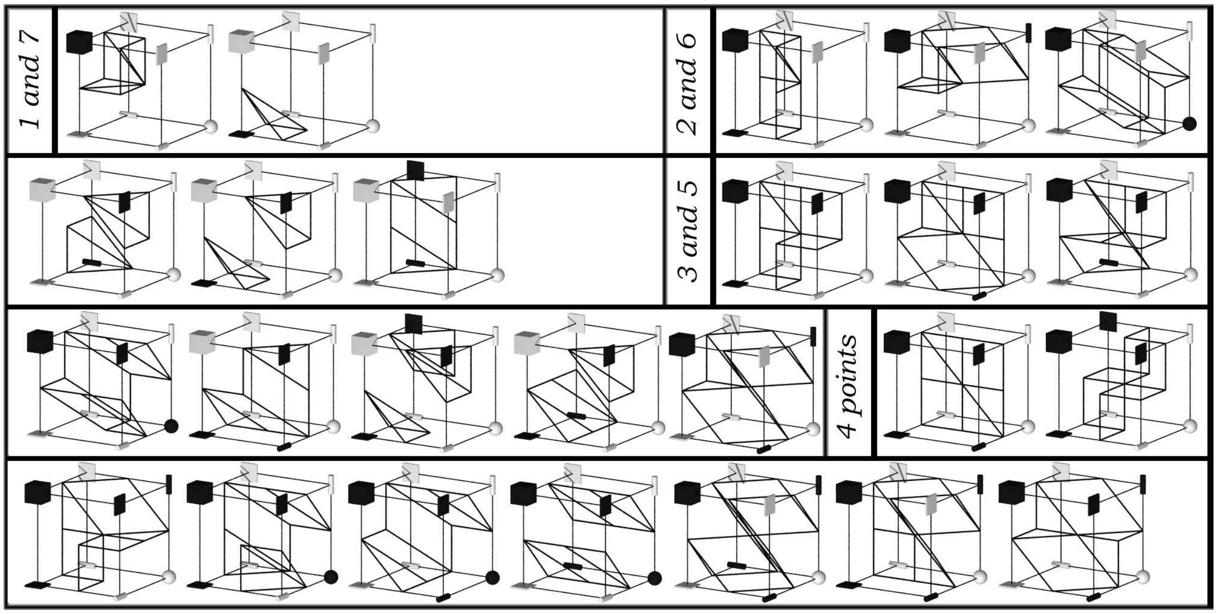



**Fig. 13.**
Configurations obtained for the look-up table of the Marching Cubes-like algorithm in the $H^3$ case. Whenever several configurations are identical up to rotations and symmetries, only one is presented here. While the original Marching-Cube Algorithm generates from 1 to 4 triangles for each configuration, the count here range from 2 to 12 triangles (2 to 6 frontier orders elements, yet some are embedded as parallelograms).

and that the surface still separates the object from the background.

The simplification process, illustrated in Fig. 14, is as follows: the configurations are first triangulated (at worst, the chain complex of the frontier order can be used for this purpose), then stellar moves [3] are applied to reduce the number of faces. In particular, in order to ensure the coherence of the frontier between adjacent unit cubes (in dimension 3), whenever the intersection of a cube face with the triangulated embedded frontier order $\mathcal{F}$ contains a point $A$ belonging to two 1-simplexes (*i.e.*, segments) $\{A, B\}$ and $\{A, C\}$ we apply the stellar weld $(\{B, C\}, \{A\})^{-1}_{\mathcal{F}}$. This move, illustrated in Fig. 14.b-c, replaces both $\{A, B\}$ and $\{A, C\}$ by the 1-simplex $\{B, C\}$ (it also replaces appropriately any other simplex in which $A$ was included).
It should be noted that, while the present article does not provide any proof that stellar moves applied to an $n$-surface result in an $n$-surface, it is well known that such moves preserve manifold properties. Thus, this method can at least be used safely in dimension 3 (see Th. 17), resulting in the look-up table depicted in Fig. 15.
We depict in Fig. 16 the surface obtained by our Marching Cubes-like algorithm, using the original look-up table in Fig. 16.a and the simplified look-up table in Fig. 16.b. The original data and the method used to obtain them are described in [31].

Similarly to the original Marching Cubes algorithm, the complexity of our algorithm is in $O(N \times 2^n)$, where $N$ is the size (number of object and background points) of the input data and $n$ is the dimension of the space. Some implementation details (including the lookup tables) may be found at the address: www.esiee.fr/∼info/xavier/work/MC04.html

## CONCLUSION

We presented new results on $n$-surfaces and frontier orders. In particular, we obtained a classification theorem (Th. 17) which shows that, in the framework of simplicial complexes, $n$-surfaces are particular cases of pseudomanifolds, and that combinatorial manifolds are particular cases of $n$-surfaces. An important consequence of this theorem is that $n$-surfaces verify a discrete analog of the Jordan-Brouwer theorem, for any $n$.
We also proved that the frontier order of any object in an $n$-dimensional "regular" space is an $(n-1)$-surface, and

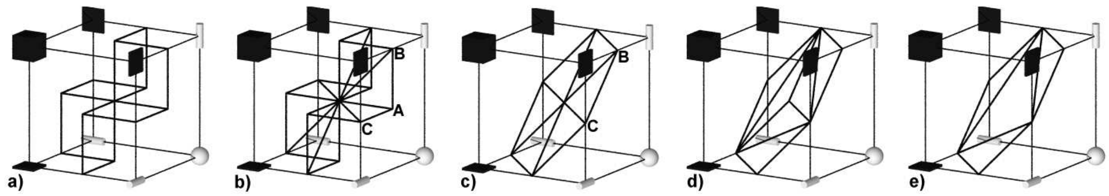



**Fig. 14.** a) is an original configuration. b) is a triangulation of a). c) is obtained from b) by the anti-stellar move replacing the vertex $A$ by the 1-simplex $\{B, C\}$, this same move being applied to all points located on the centers of the faces (this is the important part since this move has effect not only on this cube but on the neighboring ones as well). d) and e) are then obtained by consecutive stellar moves.

presented an application of this result to the design of topologically sound Marching Cubes-like algorithms.
In another article [35], we use some of the properties of this paper to show the strong link between frontier orders and the notion of regular neighborhood, as known in the framework of piecewise linear topology.

## APPENDIX: Proofs of secondary properties

**Proof of Prop. 6**
*No proper sub-order of an $n$-surface is an $(n+k)$-surface, $n, k \geq 0$.*
The case $k > 0$ is an immediate consequence of Prop. 3, suppose now $k = 0$.
The property is obvious for $n = 0$, assume that it is true for a given $n \geq 0$.
Let $|Y|$ be an $(n+1)$-surface which is a sub-order of an $(n+1)$-surface $|X|$. For any $y$ in $Y$, $|\theta^{\square}_Y(y)|$ and $|\theta^{\square}_X(y)|$ are $n$-surfaces and $\theta^{\square}_Y(y) \subseteq \theta^{\square}_X(y)$. Thus, by recurrence hypothesis, we have $\theta^{\square}_Y(y) = \theta^{\square}_X(y)$.
Since $|X|$ is connected, we easily deduce that $|Y| = |X|$. □

**Lemma A**
*Let $|X|$ be an order. Let $Y \subseteq X$ and let $z \in Y$. Then $|Y \cap \theta^{\square}_X(z)| = |\theta^{\square}_Y(z)|$.*
By definition of an induced sub-order, it is sufficient to prove that $Y \cap \theta^{\square}_X(z) = \theta^{\square}_Y(z)$. We have:

$$\begin{aligned}\theta^{\square}_Y(z) &= \{y \in Y, (z,y) \in \theta^{\square}_X \cap (Y \times Y)\}\\ &= \{y \in X, (z,y) \in \theta^{\square}_X\} \cap Y\\ &= \theta^{\square}_X(z) \cap Y\end{aligned}$$

□

**Proof of Prop. 7**
*Let $|X|$ be an $n$-surface, $n \geq 1$, and let $Y \subset X$. If $|Y|$ is a $k$-surface, $k \geq 0$, and if $|Y|$ separates $|X|$, then we have necessarily $k = n-1$.*
The property is trivially true for $n = 1$.
Then, considering that the property is true for $n-1$, let us suppose that $|X|$ is an $n$-surface and that $|Y|$ is a $k$-surface which separates $|X|$, with $0 \leq k < n-1$. Consequently, there are two points $x$ and $y$ in $X \setminus Y$ and a maximal integer $i > 0$ such that :
(1) No path from $x$ to $y$ in $|X|$ contains less than $i$ elements of $Y$.
Considering a path $(x = z_0, \ldots, z_m = y)$ containing exactly $i$ elements of $Y$, $x \in X \setminus Y$ implies that some $j > 0$ exists such that $\{x = z_0, \ldots, z_{j-1}\} \subseteq X \setminus Y$ while $z_j \in Y$. The $\theta^{\square}$-neighborhood of $z_j$ in $|X|$ is an $(n-1)$-surface and its neighborhood in $|Y|$ is a $(k-1)$-surface, consequently, by induction hypothesis, the neighborhood of $z_j$ in $|X \setminus Y|$ is connected. Moreover, by lemma A, we see that $|\theta^{\square}_X(z_j) \cap \theta^{\square}_X(z_{j+1})| = |\theta^{\square}_Z(z_{j+1})|$ with $Z = \theta^{\square}_X(z_j)$. Since $|Z|$ is an (n-1)-surface, we deduce that $|\theta^{\square}_X(z_j) \cap \theta^{\square}_X(z_{j+1})|$ is an $(n-2)$-surface, and thus is not a proper sub-order of $|Y|$ (Prop. 6). Furthermore $|\theta^{\square}_X(z_j) \cap \theta^{\square}_X(z_{j+1})| \neq |Y|$ since $z_j$ belongs to $Y$ but not to $\theta^{\square}_X(z_j)$. Thus, there exists a path from $z_{j-1}$ to $z_{j'}$, with $z_{j'} \in \theta^{\square}_X(z_{j+1})$ in $|X \setminus Y|$, and, consequently, a path from $x$ to $y$ in $|X|$ containing $i-1$ elements of $Y$, which is in contradiction with (1). □

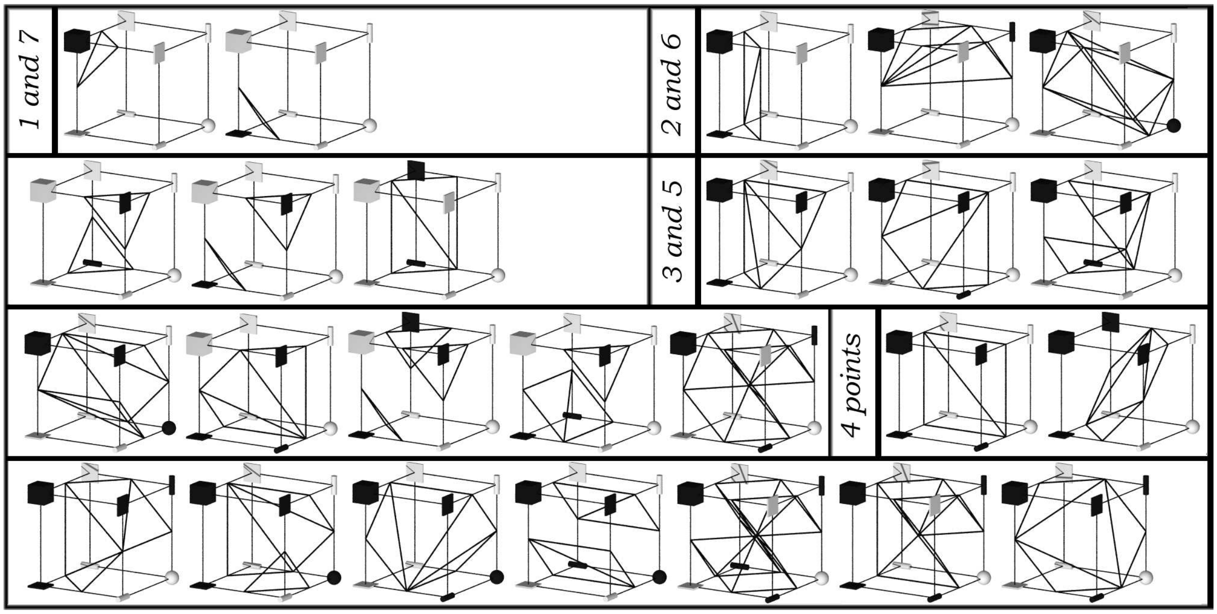



**Fig. 15.** Simplified configurations obtained for look-up table of the Marching Cubes-like algorithm in case of $H^3$, derived from the configurations presented in figure 13. It should be noticed that different frontiers may have identical simplifications. Most simplified configurations are equivalent to the corresponding configuration of the original Marching-Cubes algorithm; that is, they have the same number of triangles, the same intersection with the cube boundary and are stellar equivalent. Nevertheless some new configurations appear whenever two points located on the opposite corner of a face or cube are adjacent according to $|H^3|$ topology; and one of the original algorithm configurations, assuming four non-adjacent corners, has no equivalent here.

**Proof of Prop. 8**

*Let $s$ be a simplex of a simplicial complex $C$, $link(s, C) = \alpha_C(\beta_C(s)) \setminus \beta_C(\alpha_C(s))$.*

Let $t$ be a simplex in $link(s, C)$. Since $t$ is joinable with $s$ their intersection is empty, which implies $t \notin \beta_C(\alpha_C(s))$ (*i.e.* $t$ does not contain any subset of $s$). Then, since $s \circ t = s \cup t$, we can deduce that $s \circ t \in \beta_C(s)$ while $t \in \alpha_C(s \circ t)$. Thus $t \in \alpha_C(\beta_C((s))) \setminus \beta_C(\alpha_C(s))$.

Let $t$ be a simplex in $\alpha_C(\beta_C(s)) \setminus \beta_C(\alpha_C(s))$. Since $t \notin \beta_C(\alpha_C(s))$, $t \cap s = \emptyset$, *i.e.*, $s$ and $t$ are joinable. Then, since $t \in \alpha_C(\beta_C(s))$, there exists some element $u \in \beta_C(s)$ such that $s \subseteq u$ and $t \subseteq u$; thus $s \circ t \subseteq u$ (consequently $s \circ t \in C$) and $t \in link(s, C)$. □

**Proof of Prop. 9**

*Let $s$ be a simplex of a simplicial complex $C$, $link(s, C)$ is (order) isomorphic to $\beta_C^\square(s)$.*

Let us consider the transformation $\Phi_s$ which associates to any element $u$ of $\beta_C^\square(s)$ the element $t = u \setminus s$ of $link(s, C)$. It is obvious that $\Phi_s$ is a bijection from $\beta_C^\square(s)$ to $link(s, C)$ which preserves the inclusion. □

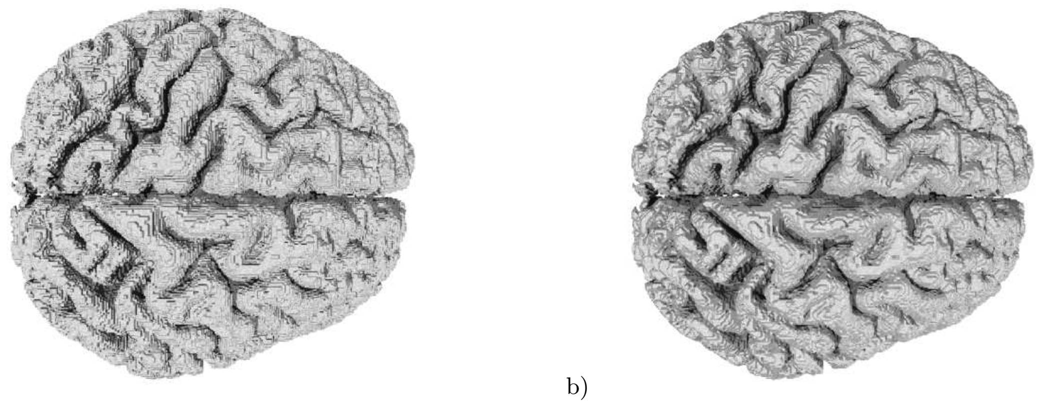
**Proof of Prop. 10**

*Let $s$ be an $n$-simplex, with $n \geq 0$, then $\alpha^\square(s)$ (*i.e., *the boundary of $s$) is an $(n-1)$-surface.*

The property is obviously true for $n = 0, 1$. Let us now assume that the property is true for all $i \leq n$, with $n > 0$. Let $s$ be an $(n+1)$-simplex and let $C = \alpha^\square(s)$.

Let $t$ be a $k$-simplex of $C$, $0 \leq k \leq n$. By induction hypothesis $|\alpha_C^\square(t)|$ is a $(k-1)$-surface. Remind that $\beta_C^\square(t)$ is the set of all (strict) subsets of $s$ including (strictly) $t$. It can be easily seen that the transformation $\Phi_t$ which associates to any element $u$ of $\beta_C^\square(t)$ the element $u \setminus t$ of $\alpha_C^\square(s \setminus t)$ is an isomorphism between the orders $|\beta_C^\square(t)|$ and $|\alpha_C^\square(s \setminus t)|$. Moreover, $|\alpha_C^\square(s \setminus t)|$ is an $(n-k-1)$-surface by induction hypothesis.

By Th. 2, $|\theta_C^\square(t)| = |\beta_C^\square(t)| * |\alpha_C^\square(t)|$ is an $(n-1)$-surface. Thus, since $C$ is obviously connected, $C = \alpha^\square(s)$ is an $n$-surface. □



a) b)

**Fig. 16.** Results for a segmented cortex (in $|H^3|$), a) using initial configurations b) using simplified configurations.

**Proof of Prop. 11**
*Let $C$ be a connected simplicial complex of support $\Lambda$. The simplicial complex $C$ is an $n$-surface, with $n > 0$, if, and only if, $\forall x \in \Lambda$, $link(\{x\}, C)$ is an $(n-1)$-surface.*
Let us first remind that, for any 0-simplex $s$ of $C$, the link of $s$ in $C$ is isomorphic to $\beta^\square_C(s)$ (Prop. 9). The direct implication is obvious, let us prove the converse.
By Cor. 5, applied to $|\theta^\square_C(\{x\})|$, $C$ is an $n$-complex. Let $s$ be a $k$-simplex of $C$, $0 \leq k \leq n$. We will proceed by induction on $k$ to prove that $|\theta^\square_C(s)|$ is an $(n-1)$-surface.

- By Prop. 10, $|\alpha^\square_C(s)|$ is a $(k-1)$-surface.
- By hypothesis (and Prop. 9), if $k = 0$ then $|\beta^\square_C(s)| = |\theta^\square_C(s)|$ is an $(n-1)$-surface.
- Let us now suppose that $k > 0$ and that, for some $(k-1)$-simplex $s_{k-1}$ of $C$ in $\theta^\square_C(s)$, $|X| = |\beta^\square_C(s_{k-1})|$ is an $(n-k)$-surface. Then $|\beta^\square_C(s)| = |\theta^\square_X(s)|$ is an $(n-k-1)$-surface.

Thus, by Th. 2, $|\theta^\square_C(s)|$ is an $(n-1)$-surface for any $s$ and, $C$ being connected, $C$ is an $n$-surface. □

**Proof of Prop. 13**
*Let $|X|$ and $|Y|$ be two orders such that $X \cap Y = \emptyset$, $\mathcal{C}^{X*Y} = \mathcal{C}^{Y*X} = \mathcal{C}^X \circ \mathcal{C}^Y$.*
Obviously, we have $\mathcal{C}^{X*Y} = \mathcal{C}^{Y*X}$. By definition:

$\mathcal{C}^{X*Y} = \{\{s_0, \ldots, s_k\} \subseteq X * Y, \forall i, j \in [0, k], s_i \in \theta_{X*Y}(s_j)\}$

and:

$\mathcal{C}^X \circ \mathcal{C}^Y = \mathcal{C}^X \cup \mathcal{C}^Y \cup \{\{x_0, \ldots, x_{i-1}, y_i, \ldots, y_k\}, \{x_0, \ldots, x_{i-1}\} \in \mathcal{C}^X, \{y_i, \ldots, y_k\} \in \mathcal{C}^Y\}$

Let $s_i \in \theta_{X*Y}(s_j)$. We have three possible cases:

- either $s_i \in X$, $s_j \in X$ and $s_i \in \theta_X(s_j)$
- or $s_i \in Y$, $s_j \in Y$ and $s_i \in \theta_Y(s_j)$
- or $\{s_i, s_j\} = \{x, y\}, x \in X, y \in Y$.

Consequently, any element of $\mathcal{C}^{X*Y}$ is either a chain of $|X|$ (if it is composed uniquely of elements of $|X|$), or a chain of $|Y|$ (if it is composed uniquely of elements of $|Y|$) or the union of a chain of $|X|$ and a chain of $|Y|$. Conversely any chain of $|X|$ or $|Y|$ obviously belongs to $\mathcal{C}^{X*Y}$, as well as the join of any chain of $|X|$ with any chain of $|Y|$. □

**Proof of Prop. 20**
*Let $C$ be a simplicial complex and let $X$ be its support. Let $K$ and $L$ be two unrelated subsets of $X$. Then $C_{K/\overline{K}}$ and $C_{L/\overline{L}}$ are mutually disconnected.*
Let $K$ and $L$ be two unrelated (and thus disjoint) subsets of $C$ and let us suppose that $C_{K/\overline{K}}$ is adjacent to $C_{L/\overline{L}}$. Let $s$ be a simplex of $C_{K/\overline{K}}$ and let $t$ be a simplex of $C_{L/\overline{L}}$ such that $s \in \theta_C(t)$. Without loss of generality, we can assume that $s \in \alpha_C(t)$ . Since $s \in C_{K/\overline{K}}$, we know that $s \cap K \neq \emptyset$ and, consequently $t \cap K \neq \emptyset$. Since $t \in C_{L/\overline{L}}$, there exists a simplex $\{t_L, t_K\} \subseteq t$, with $t_L \in L$ and $t_K \in K$. Then, either $t_L = t_K$ implying $K$ and $L$ are not even

disjoint, or $\{t_L, t_K\}$ is a 1-simplex in which case $K$ and $S$ are not unrelated. □

## References


1. Hudson, J.: Piecewise Linear Topology. W.A. Benjamin inc. (1969)
2. Alexander, J.: The combinatorial theory of complexes. Ann. Math. **31** (1930) 294–322
3. Lickorish, W.: Simplicial moves on complexes and manifolds. Geometry and Topology Monograph, Proccedings of the KirbyFest **2** (1998) 229–320
4. Moise, E.: Affine structures on 3-manifolds. Annals of mathematics **56** (1952) 96–114
5. Kong, T.Y., Rosenfeld, A.: Digital topology: Introduction and survey. Computer Vision, Graphics and Image Processing **48** (1989) 357–393
6. Morgenthaler, D., Rosenfeld, A.: Surfaces in three-dimensional images. Information and Control **51** (1981) 227–247
7. Malgouyres, R.: A definition of surfaces of $\mathbb{Z}^3$. In: Discrete Geometry for Computer Imagery. (1993) 23–34
8. Couprie, M., Bertrand, G.: Simplicity surfaces : a new definition of surfaces in $\mathbb{Z}^3$. In: SPIE Vision Geometry V Proceedings. Volume 3454. (1998) 40–51
9. Ayala, R., Dominguez, E., Frances, A., Quintero, A., Rubio, J.: On surfaces in digital topolgy. In: Discrete Geometry for Computer Imagery. (1995) 271–276
10. Bertrand, G., Couprie, M.: A model for digital topology. In Springer, ed.: Discrete Geometry for Computer Imagery. Volume 1568 of LNCS. (1999) 229–241
11. Khalimsky, E.: On topologies of generalized segments. Soviet Mat. Doklady **10** (1969) 1508–1511
12. Bertrand, G.: New notions for discrete topology. In Springer, ed.: Discrete Geometry for Computer Imagery. Volume 1568 of LNCS. (1999) 218–228
13. Alexandroff, P.: Combinatorial topology. Dover Publications (1947)
14. Evako, A.V., Kopperman, R., Mukhin, Y.V.: Dimensional properties of graphs and digital spaces. Jour. of Math. Imaging and Vision **6** (1996) 109–119
15. Ivashchenko, A.V.: Representation of smooth surfaces by graphs. transformations of graphs which do not change the Euler characteristic of graphs. Discrete Mathematics **122** (1993) 219–133
16. Ivashchenko, A.V.: Dimension on discrete spaces. International Journal of Theoretical Physics (1994)
17. E.D. Khalimsky, R. Kopperman, P.M.: Computer graphics and connected topologies on finite ordered sets. Topology Appl. **36** (1990) 1–17
18. R. Kopperman, P.R. Meyer, R.W.: A Jordan surface theorem for three-dimensional digital spaces. Discrete Computational Geometry **6** (1991) 155–161
19. Herman, G.: Discrete multidimensional Jordan surfaces. Graphicals Models and Image Processing **54** (1992) 507–515
20. Herman, G.: Oriented surfaces in digital spaces. Graphicals Models and Image Processing **55** (1993) 381–396
21. Udupa, J.: Multidimensional digital boundaries. Graphicals Models and Image Processing **56** (1994) 311–323
22. R. Aharoni, G.T. Herman, M.L.: Jordan graphs. Graphicals Models and Image Processing **58** (1996) 345–359
23. J. Burguet, R.M.: Strong thinning and polyhedric approximation of the surface of a voxel object. Discrete Applied Mathematics **125** (2003) 93–114
24. Lorensen, W., Cline, H.: Marching cubes: a high resolution 3D surface construction algorithm. Computer Graphics **21** (1987) 163–169
25. Lachaud, J.O., Montanvert, A.: Continuous analogs of digital boundaries: A topological approach to iso-surfaces. Graphical models **62** (2000) 129–164
26. Daragon, X., Couprie, M., Bertrand, G.: Marching chains algorithm for Alexandroff-Khalimsky spaces. In: SPIE Vision Geometry XI proceedings. (2002) 51–62
27. Daragon, X., Couprie, M., Bertrand, G.: Discrete frontiers. In Springer, ed.: Discrete Geometry for Computer Imagery. Volume 2886 of LNCS. (2003) 236–245
28. Kovalevsky, V.: Finite topology as applied to image analysis. Computer Vision, Graphics and Image Processing (1989)
29. Cointepas, Y., Bloch, I., Garnero, L.: A cellular model for multi-objects multi-dimensional homotopic deformations. Pattern Recognition **34(9)** (2001) 1785–1798
30. Lohou, C., Bertrand, G.: Poset approach to 3d parallel thinning. In: SPIE Vision Geometry VIII. Volume 3811. (1999) 45–56
31. Daragon, X., Couprie, M.: Segmentation topologique du neo-cortex cérébral depuis des données IRM. In: RFIA 2002. Volume 3. (2002) 809–818
32. Pescatore, J.: Maillages homotopiques tétraédriques des tissus de la tête pour le calcul du problème direct en électro/magnéto-encéphalographie. Ph.D. Thesis, ENST Paris (2001)
33. J. Burguet, I.B.: Homotopic labeling of elements in a tetrahedral mesh for the head modeling. In: Procs. 9th Iberoamerican Congress on Pattern Recognition (to appear). (2004)
34. Couprie, M., Bertrand, G., Kenmochi, Y.: Discretization in 2d and 3d orders. Graphical Models **65** (2003) 77–91
35. Daragon, X., Couprie, M., Bertrand, G.: Frontier orders and derived neighborhood. To appear in Discrete Applied Mathematics (2004)