Find out the difference(s)?

Shape 1  |  Shape 2

J. Cousty : MorphoGraph and Imagery 1/26
Find out the difference(s)?

Graph $G_1$

Graph $G_2$
Connectivity in graphs

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MorphoGraph and Imagery

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Outline of the lecture

1. Path
2. Connectivity
3. Algorithms
4. Degrees
Definition

- Let $G = (E, \Gamma)$ be a graph, and let $x$ and $y$ be two vertices in $E$.
- A path from $x$ to $y$ (in $G$) is a sequence $\pi = (x_0, \ldots, x_\ell)$ of vertices in $E$ such that:
  - $\forall i \in \{1, \ldots, \ell\}$, $x_i \in \Gamma(x_{i-1})$.
  - $x_0 = x$ and $x_\ell = y$. 

Exemple

$\pi = (1, 2, 3)$ is a path of length 2.
Path

Definition

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  - $\forall i \in \{1, \ldots, \ell\}$, $x_i \in \Gamma(x_{i-1})$
  - $x_0 = x$ and $x_\ell = y$
- If $\pi = (x_0, \ldots, x_\ell)$ is a path, $\ell$ is called its length.

Exemple

$\pi = (1, 2, 3)$ is a path of length 2.
Some remarkable paths

- A path of length 0 is called a *trivial path*

Exemple

- (3) *is a trivial path*
Some remarkable paths

- A path of length 0 is called a *trivial path*.
- A non-trivial path $\pi = (x_0, \ldots, x_\ell)$ is called a *circuit* if $x_0 = x_\ell$.

**Exemple**

- (3) is a trivial path.
- (1, 2, 3, 1) is a circuit.
Some remarkable paths

- A path of length 0 is called a **trivial path**
- A non-trivial path \( \pi = (x_0, \ldots, x_\ell) \) is called a **circuit** if \( x_0 = x_\ell \)
- A path \( \pi = (x_0, \ldots, x_\ell) \) is called **elementary** if any two of its vertices are distinct (except possibly \( x_0 \) and \( x_\ell \)): \( \forall i, j \in \{0, \ldots, \ell\}, \ i \neq j \implies x_i \neq x_j \) (where \( \{i, j\} \neq \{0, \ell\} \))

**Exemple**

- \((3)\) is a trivial path
- \((1, 2, 3, 1)\) is a circuit that is elementary
Some remarkable paths

- A path of length 0 is called a **trivial path**
- A non-trivial path $\pi = (x_0, \ldots, x_\ell)$ is called a **circuit** if $x_0 = x_\ell$
- A path $\pi = (x_0, \ldots, x_\ell)$ is called **elementary** if any two of its vertices are distinct (except possibly $x_0$ and $x_\ell$): $\forall i, j \in \{0, \ldots, \ell\}$, $i \neq j \implies x_i \neq x_j$ (where $\{i, j\} \neq \{0, \ell\}$)

**Exemple**

- (3) is a trivial path
- (1, 2, 3, 1) is a circuit that is elementary
- (1, 3, 1, 2, 3, 1) is a circuit that is not elementary
Basic properties

Property

- Any path \( \pi \) from \( x \) to \( y \) contains an elementary path from \( x \) to \( y \)
Basic properties

- Any path $\pi$ from $x$ to $y$ contains an elementary path from $x$ to $y$
- The length of an elementary circuit is less than $n$ (where $n = |E|$)
Basic properties

- Any path $\pi$ from $x$ to $y$ contains an elementary path from $x$ to $y$
- The length of an elementary circuit is less than $n$ (where $n = |E|$)
- The length of an elementary path which is not a circuit is less than $n - 1$
Definition

Let $G_s = (E, \Gamma_s)$ be the symmetric closure of $G$
Undirected paths and cycles

**Definition**

- Let $G_s = (E, \Gamma_s)$ be the symmetric closure of $G$
- Any path in $G_s$ is called an undirected path (in $G$)
**Definition**

- Let \( G_s = (E, \Gamma_s) \) be the symmetric closure of \( G \)
- Any path in \( G_s \) is called an **undirected path** (in \( G \))
- A **cycle** (in \( G \)) is a circuit in \( G_s \) that does not pass twice by the same edge
**Undirected paths and cycles**

**Definition**

- Let $G_s = (E, \Gamma_s)$ be the symmetric closure of $G$.
- Any path in $G_s$ is called an **undirected path** (in $G$).
- A cycle (in $G$) is a circuit in $G_s$ that does not pass twice by the same edge.

**Remark.** If $\pi = (x_0, \ldots, x_\ell)$ is an undirected path, then $\pi' = (x_\ell, \ldots, x_0)$ is an undirected path (since $(E, \Gamma_s)$ is a symmetric graph, which implies that $x_i \in \Gamma_s(x_{i-1}) \iff x_{i-1} \in \Gamma_s(x_i)$).
Illustration: path and undirected path, circuit and cycles

- (1, 2, 1) is not a path
- (1, 2, 3, 1, 2, 4) is a path
- (4, 2, 1) is an undirected path
- (1, 3, 2, 1) is not a circuit
- (1, 3, 1) is not a cycle
- (1, 2, 3) is a path
- (4, 2, 1) is not a path
- (1, 2, 3, 1) is a circuit
- (1, 3, 2, 1) is a cycle
Connected component

**Definition**

- Let $x \in E$. The connected component of $G$ containing $x$ is the subset $C_x$ of $E$ defined by:
  - $C_x = \{ y \in E \mid \text{there exists an undirected path from } x \text{ to } y \}$
Connected component

Definition

- Let $x \in E$. The connected component of $G$ containing $x$ is the subset $C_x$ of $E$ defined by:
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Exemple

- $C_1 = \{1, 2, 3, 4\} = C_2 = C_3 = C_4$
- $C_5 = \{5\}$
- $C_6 = C_7 = C_8 = \{6, 7, 8\}$
Connected component as equivalence classes

**Property**

1. \( \forall x \in E, \ x \in C_x \) (*reflexivity*)
2. \( \forall x, y \in E, \ y \in C_x \implies x \in C_y \) (*symmetry*)
3. \( \forall x, y, z \in E, \ [y \in C_x \text{ and } z \in C_y] \implies z \in C_x \) (*transitivity*)

**Proof.**

1. \( \forall x \in E, \ (x) \) is a (trivial) undirected path, thus \( x \in C_x \)
2. \( y \in C_x \implies \exists \text{ an undirected path } \pi = (x_0, \ldots, x_\ell) \text{ from } x \text{ to } y \implies \pi' = (x_\ell, \ldots, x_0) \text{ is an undirected path from } y \text{ to } x \implies x \in C_y \)
3. \( [y \in C_x \text{ and } z \in C_y] \implies [\exists \text{ an undirected path } \pi = (x_0, \ldots, x_\ell) \text{ from } x \text{ to } y \text{ and } \exists \text{ an undirected path } \pi' = (y_0, \ldots, y_m) \text{ from } y \text{ to } z] \implies \pi'' = (x_0, \ldots, x_\ell, y_1, \ldots, y_m) \text{ is an undirected path from } x \text{ to } z \implies z \in C_x \)
Strongly connected component

Definition

- Let $x \in E$. The strongly connected component (of $G$) containing $x$ is the subset $C'_x$ of $E$ defined by

$$C'_x = \{y \in E \mid \exists \text{ a path from } x \text{ to } y \text{ and } \exists \text{ a path from } y \text{ to } x\}$$
Strongly connected component

**Definition**

- Let $x \in E$. The strongly connected component (of $G$) containing $x$ is the subset $C'_x$ of $E$ defined by
  - $C'_x = \{y \in E \mid \exists$ a path from $x$ to $y$ and $\exists$ a path from $y$ to $x\}$

**Exemple**

- $C'_1 = \{1, 2, 3\} = C'_2 = C'_3$
- $C'_4 = \{4\} ; C'_5 = \{5\}$
- $C'_6 = \{6\} ; C'_7 = \{7\}$
- $C'_8 = \{8\}$
**Exercise.** Are the strongly connected components of a graphs equivalence classes? Proof or counter-example?
Connected and strongly connected components computation

Method

- We will first study a characterization of connected components and of strongly connected components based on morphological dilation.
- This will allow us to propose efficient algorithm to compute them.
Iterated operators

Definition

- Let $\gamma$ be an operator and $i \in \mathbb{N}$
- We denote by $\gamma^i$ the operator defined by
  1. $\gamma^i = \gamma \gamma^{i-1}$
  2. $\gamma^0 = \text{Id}$ (i.e. $\forall X \subseteq E, \gamma^0(X) = X$)
Iterated operators

Definition

- Let $\gamma$ be an operator and $i \in \mathbb{N}$
- We denote by $\gamma^i$ the operator defined by
  1. $\gamma^i = \gamma \gamma^{i-1}$
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Exemple (iterated dilation)

1. $\delta^0_\Gamma(X) = X$
2. $\delta^1_\Gamma(X) = \delta_\Gamma(\delta^0_\Gamma(X)) = \delta_\Gamma(X)$
3. $\delta^2_\Gamma(X) = \delta_\Gamma(\delta^1_\Gamma(X)) = \delta_\Gamma(\delta_\Gamma(X))$
4. $\delta^3_\Gamma(X) = \delta_\Gamma(\delta^2_\Gamma(X)) = \delta_\Gamma(\delta_\Gamma(\delta_\Gamma(X)))$
5. $\vdots$
6. $\delta^i_\Gamma(X) = \delta_\Gamma(\delta^{i-1}_\Gamma(X)) = \delta_\Gamma(\ldots \delta_\Gamma(X) \ldots)$
Illustration: iterated dilation

Exemple

\[ X = \delta_1^0(X) = \{1\} \]
Exemple

- $X = \delta^0_\Gamma(X) = \{1\}$
- $\delta^1_\Gamma(X) = \{2\}$
Exemple

- $X = \delta^{0}_{\Gamma}(X) = \{1\}$
- $\delta^{1}_{\Gamma}(X) = \{2\}$
- $\delta^{2}_{\Gamma}(X) = \{3, 4\}$
Property

- Let $x \in E$ and $i \in \mathbb{N}$.
- The two following equalities hold true:
  - $\delta_i^i(\{x\}) = \{y \in E \mid \exists \text{ a path from } x \text{ to } y \text{ of length } i\}$
  - $\delta_{i-1}(\{x\}) = \{y \in E \mid \exists \text{ a path from } y \text{ to } x \text{ of length } i\}$
Iterated dilation and paths of given length

Property

- Let $x \in E$ and $i \in \mathbb{N}$
- The two following equalities hold true
  - $\delta^i_\Gamma(\{x\}) = \{y \in E \mid \exists \text{ a path from } x \text{ to } y \text{ of length } i\}$
  - $\delta^{i-1}_\Gamma(\{x\}) = \{y \in E \mid \exists \text{ a path from } y \text{ to } x \text{ of length } i\}$

Definition

- We define for any $X \subseteq E$ and any $p \in \mathbb{N}$
  - $\hat{\delta}^p(\Gamma) = \bigcup \{\delta^i(X) \mid i \in \{0, \ldots, p\}\}$
Iterated dilation and paths of given length

Property

- Let $x \in E$ and $i \in \mathbb{N}$
- The two following equalities hold true
  - $\delta_{i}^\Gamma(\{x\}) = \{y \in E \mid \exists \text{ a path from } x \text{ to } y \text{ of length } i\}$
  - $\delta_{i-1}^\Gamma(\{x\}) = \{y \in E \mid \exists \text{ a path from } y \text{ to } x \text{ of length } i\}$

Definition

- We define for any $X \subseteq E$ and any $p \in \mathbb{N}$
  - $\hat{\delta}_{p}^\Gamma(X) = \bigcup \{\delta_{i}^\Gamma(X) \mid i \in \{0, \ldots, p\}\}$
- The set $\hat{\delta}_{p}^\Gamma(X)$ is called the pre-closure of $X$ of rank $p$
Corollary

- Let $x \in E$
- The two following equalities hold true
  - $\delta_p^\Gamma(\{x\}) = \{y \in E \mid \exists \text{ a path from } x \text{ to } y \text{ of length } \leq p\}$
  - $\delta_{\Gamma-1}(\{x\}) = \{y \in E \mid \exists \text{ a path from } y \text{ to } x \text{ of length } \leq p\}$
Transitive closure and paths

Corollary

- Let \( x \in E \)
- The two following equalities hold true
  - \( \hat{\delta}^p_\Gamma(\{x\}) = \{ y \in E \mid \exists \text{ a path from } x \text{ to } y \text{ of length } \leq p \} \)
  - \( \hat{\delta}^{p-1}_\Gamma(\{x\}) = \{ y \in E \mid \exists \text{ a path from } y \text{ to } x \text{ of length } \leq p \} \)

Definition

- Let \( x \in E \), the (transitive) closure of \( \{x\} \) is the set
  - \( \hat{\delta}_\Gamma^\infty(\{x\}) = \{ y \in E \mid \exists \text{ a path from } x \text{ to } y \} \)
Corollary

- Let $x \in E$
- The two following equalities hold true
  - $\delta_p^\Gamma(\{x\}) = \{ y \in E \mid \exists \text{ a path from } x \text{ to } y \text{ of length } \leq p \}$
  - $\delta_{\Gamma^{-1}}^p(\{x\}) = \{ y \in E \mid \exists \text{ a path from } y \text{ to } x \text{ of length } \leq p \}$

Definition

- Let $x \in E$, the (transitive) closure of $\{x\}$ is the set
  - $\delta_\Gamma^\infty(\{x\}) = \{ y \in E \mid \exists \text{ a path from } x \text{ to } y \}$

Exercise. Prove that $\delta_\Gamma^\infty(\{x\}) = \delta_{\Gamma^{-1}}^{n-1}(\{x\})$ (where $n = |E|$)
Illustration: transitive closure

Exemple

\[ X = \delta_\Gamma^0(X) = \{1\} \]

\[ X = \hat{\delta}_\Gamma^0(X) = \{1\} \]
Illustration: transitive closure

Exemple

- $X = \delta^0_\Gamma(X) = \{1\}$
- $\delta^1_\Gamma(X) = \{2\}$
- $X = \hat{\delta}^0_\Gamma(X) = \{1\}$
- $\hat{\delta}^1_\Gamma(X) = \{1, 2\}$
Illustration: transitive closure

Exemple

- $X = \delta^0_\Gamma(X) = \{1\}$
- $\delta^1_\Gamma(X) = \{2\}$
- $\delta^2_\Gamma(X) = \{3, 4\}$
- $X = \hat{\delta}^0_\Gamma(X) = \{1\}$
- $\hat{\delta}^1_\Gamma(X) = \{1, 2\}$
- $\hat{\delta}^2_\Gamma(X) = \{1, 2, 3, 4\} = \delta^7_\Gamma(X)$
Property

Let \( x \in E \). Let \( C_x \) and \( C'_x \) be respectively the connected components and the strongly connected components containing \( x \).
Transitive closure and connected components

**Property**

- Let \( x \in E \). Let \( C_x \) and \( C'_x \) be respectively the connected components and the strongly connected components containing \( x \).

- The two following equalities hold true:
  - \( C'_x = \delta_{\Gamma}^{-1}(\{x\}) \cap \delta_{\Gamma-1}^{-1}(\{x\}) \)
  - \( C_x = \delta_{\Gamma_s}^{-1}(\{x\}) = \delta_{\Gamma_s}^{-1}(\{x\}) \)

- Where \( n = |E| \) and \( \Gamma_s \) is the symmetric closure of \( \Gamma \)
Transitive closure and connected components

Property

Let $x \in E$. Let $C_x$ and $C'_x$ be respectively the connected components and the strongly connected components containing $x$.

The two following equalities hold true:

- $C'_x = \delta_{\Gamma}^{n-1}(\{x\}) \cap \delta_{\Gamma^{-1}}^{n-1}(\{x\})$
- $C_x = \delta_{\Gamma_s}^{n-1}(\{x\}) = \delta_{\Gamma_s}^{n-1}(\{x\})$

where $n = |E|$ and $\Gamma_s$ is the symmetric closure of $\Gamma$.

Important remark

To compute the connected and strongly connected components containing $x$, it is thus sufficient to compute the transitive closure of $\{x\}$ for the graphs $(E, \Gamma)$, $(E, \Gamma^{-1})$ and $(E, \Gamma_s)$. 
Naive algorithm for the transitive closure of \( \{x\} \)

**Algorithm TRANS\_NAIV** (Data: \( (E, \Gamma), x \in E \); Results: \( Z = \delta_{\Gamma}^{n-1}(\{x\}) \))

- \( X := \{x\} \); \( Y := \emptyset \); \( Z := \{x\} \);
- **For each** \( i \) **from** 1 **to** \( n - 1 \) **do**
  - \( Y := \text{DIL}((E, \Gamma), X) \);
  - \( Z := Z \cup Y \);
  - \( X := Y; Y := \emptyset \);

Complexity

By using the algorithm DIL studied during practical session # 1

The complexity of TRANS\_NAIV is \( O(n^2 + nm) \) (where \( n = |E| \) and \( m = |\Gamma| \)).
Naive algorithm for the transitive closure of \( \{x\} \)

Algorithm TRANS\_NAIV ( Data: \((E, \Gamma), x \in E\); 
Results: \( Z = \delta_{\Gamma}^{n-1}(\{x\}) \))

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Complexity

- By using the algorithm DIL studied during practical session \# 1
- The complexity of TRANS\_NAIV is
  - \( O(n^2 + nm) \) (where \( n = |E| \) and \( m = |\Gamma| \))
Linear-time algorithm for the transitive closure of \{x\}

Algorithm TRANS (Data: \((E, \Gamma), x \in E\); Result: \(Z = \delta_{\Gamma}^{n-1}(\{x\})\))

\[
X := \{x\}; \quad Y := \emptyset; \quad Z := \{x\};
\]

For each \(i\) from 1 to \(n - 1\) do

- While \(\exists y \in X\) do
  - \(X := X \setminus \{y\}\);
  - For each \(z \in \Gamma(y)\) do
    - If \(z \notin Z\) then \(Y := Y \cup \{z\}; Z := Z \cup \{z\}\);
  - \(X := Y; Y := \emptyset\);
Linear-time algorithm for the transitive closure of \( \{x\} \)

**Algorithm TRANS**

*Data: \((E, \Gamma), x \in E\)*

*Result: \( Z = \delta_{\Gamma}^{n-1}(\{x\}) \)*

\[
X := \{x\}; Y := \emptyset; Z := \{x\}; \\
\text{For each } i \text{ from } 1 \text{ to } n-1 \text{ do} \\
\quad \text{While } \exists y \in X \text{ do} \\
\qquad X := X \setminus \{y\}; \\
\qquad \text{For each } z \in \Gamma(y) \text{ do} \\
\qquad\quad \text{If } z \notin Z \text{ then } Y := Y \cup \{z\}; Z := Z \cup \{z\}; \\
\qquad X := Y; Y := \emptyset;
\]

**Complexity**

- If \( X \) and \( Y \) are represented by LLs and if \( Z \) is represented by a BA
- The time-complexity of TRANS is linear
  - \( O(n + m) \) (where \( n = |E| \) and \( m = |\Gamma| \))
Linear-time algorithm for the transitive closure of \( \{x\} \)

Algorithm TRANS can be further simplified without changing its complexity:

**Algorithm TRANS** (Data: \((E, \Gamma), x \in E\); Result: \(Z = \delta_{\Gamma}^{-1}(\{x\})\))

- \(X := \{x\}; Z := \{x\}\);
- **While** \(\exists y \in X\) **do**
  - \(X := X \setminus \{y\}\);
  - **For each** \(z \in \Gamma(y)\) **do**
    - **If** \(z \notin Z\) **then** \(X := X \cup \{z\}; Z := Z \cup \{z\}\);
Algorithm SCC (Data: $(E, \Gamma), x \in E$; Result: $Z = C'_x$)

- $X := \text{TRANS}((E, \Gamma), x)$;
- $\Gamma^{-1} := \text{SYM}_1(E, \Gamma)$;
- $Y := \text{TRANS}((E, \Gamma^{-1}), x)$;
- $Z := X \cap Y$;
Computing the strongly connected component containing $x$

**Algorithm SCC**

**Data:** $(E, \Gamma), \ x \in E$

**Result:** $Z = C'_x$

- $X := \text{TRANS}((E, \Gamma), x)$;
- $\Gamma^{-1} := \text{SYM}_1(E, \Gamma)$;
- $Y := \text{TRANS}((E, \Gamma^{-1}), x)$;
- $Z := X \cap Y$;

**Complexity**

- **Using**
  - the linear-time algorithm $\text{SYM}_1$ (first lecture)
  - the linear-time $\text{TRANS}$
- The time-complexity of SCC is linear:
  - $O(n + m)$ (where $n = |E|$ and $m = |\Gamma|$)
Computing the connected component containing $x$

### Algorithm CC
(Data: $(E, \Gamma)$, $x \in E$; Result: $Y = C_x$)

- $\Gamma^{-1} := \text{SYM}_1(E, \Gamma)$;
- $\Gamma_s := \Gamma \cup \Gamma^{-1}$;
- $Y := \text{TRANS}((E, \Gamma_s), x)$
Computing the connected component containing $x$

### Algorithm CC

**Data:** $(E, \Gamma)$, $x \in E$

**Result:** $Y = C_x$

- $\Gamma^{-1} := \text{SYM}_1(E, \Gamma)$;
- $\Gamma_s := \Gamma \cup \Gamma^{-1}$;
- $Y := \text{TRANS}((E, \Gamma_s), x)$

### Complexity

- **Using**
  - the linear-time algorithm \text{SYM}_1 (first lecture)
  - the linear-time \text{TRANS}

- **The time-complexity of CC is linear:**
  - $O(n + m)$ (where $n = |E|$ and $m = |\Gamma|$)
Problem

- In a charity diner, is there always two people having the same number of friends who are present in the diner?
- Can we draw in the plan five distinct lines such that any of them has exactly three intersection points with the others?
Application of graphs properties

Problem

- In a charity diner, is there always two people having the same number of friends who are present in the diner?
- Can we draw in the plan five distinct lines such that any of them has exactly three intersection points with the others?

In order to answer these questions read first the two next slides!
Let $G = (E, \Gamma)$ be a graph and let $x \in E$

- The outer degree of $x$ (for $G$) is the value $d^+(x) = |\Gamma(x)|$
- The inner degree of $x$ (for $G$) is the value $d^-(x) = |\Gamma^{-1}(x)|$
- The degree of $x$ (for $G$) is the value $d(x) = d^+(x) + d^-(x)$

Let $G' = (E, \overline{\Gamma})$ be an undirected graph

- The degree of $x$ for $G'$ is the number $d(x)$ of edges that are adjacent to $x$
Prove that the two following propositions hold true

- The sum of the degrees of the vertices of a graph is even
- In any graph, there is an even number of vertices whose degree is odd