Some morphological operators in graph spaces

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Historical background

- Digital image processing
  - Transformations on the subsets of $\mathbb{Z}^2$ (binary images)
  - Transformations on the maps from $\mathbb{Z}^2$ to $\mathbb{N}$ (grayscale images)

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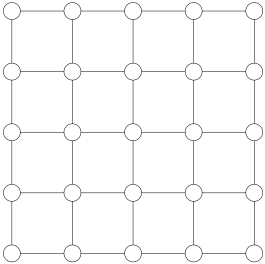
- Digital image processing
  - Transformations on the subsets of $\mathbb{Z}^2$ (binary images)
  - Transformations on the maps from $\mathbb{Z}^2$ to $\mathbb{N}$ (grayscale images)
- Mathematical morphology
  - Filtering and segmenting tools very useful in applications
  - Formally studied in lattices (*e.g.*, $2^{|E|}$)

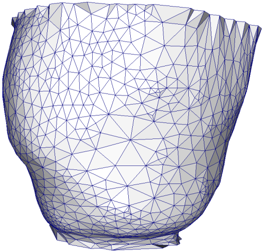
More recently

- Structured digital objects:
  - Points **and**
  - Elements between points telling how points are “glued” together
- For instance:

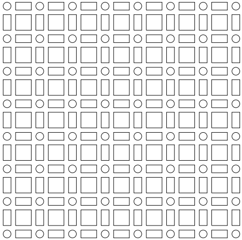
Graph



Simplicial complex



Cubical complex



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- Najman, *Ultrametric Watersheds*, ISMM (2009)
- Levillain et al., *Milena: Write Generic Morphological Algorithms Once, Run on Many kinds of Images*, ISMM (2009)

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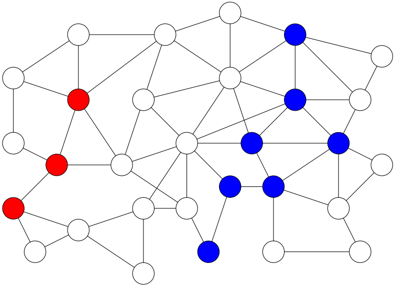

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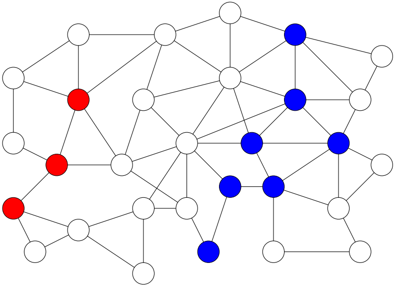

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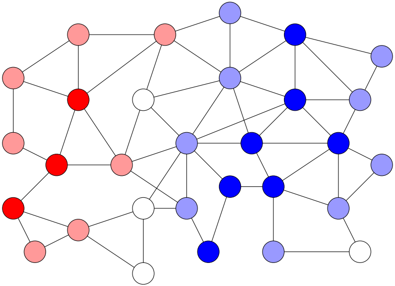

$\delta(X)$ (red & blue vertices)

Morphology in graphs

Problem

- *The workspace being a graph*
  - *What morphological operators on subsets of its vertex set?*
  - *What morphological operators on subsets of its edge set?*
  - *What morphological operators on its subgraphs?*
- *Relation between them?*

Outline

Ordering on graphs

- A *graph* is a pair $X = (X^\bullet, X^\times)$ where $X^\bullet$ is a set and $X^\times$ is composed of unordered pairs of distinct elements in $X^\bullet$

Definition

Let X and Y be two graphs.

- *If $Y^\bullet \subseteq X^\bullet$ and $Y^\times \subseteq X^\times$, then:*
  - *Y is a* subgraph *of X*
  - *we write $Y \sqsubseteq X$*
  - *we say that Y is* smaller *than X and that X is* greater *than Y*

- Hereafter, the workspace is a graph $\mathbb{G} = (\mathbb{G}^{\bullet}, \mathbb{G}^{\times})$
- We consider the famillies $\mathcal{G}^{\bullet}$, $\mathcal{G}^{\times}$ and $\mathcal{G}$ of respectively all subsets of $\mathbb{G}^{\bullet}$, all subsets of $\mathbb{G}^{\times}$ and all subgraphs of $\mathbb{G}$.

Lattice of graphs

Property

- *The set $\mathcal{G}$ of the subgraphs of $\mathbb{G}$ form a* **complete lattice**
- *The* **infimum** *and the* **supremum** *of any family $\mathcal{F} = \{X_1, \ldots X_\ell\}$ of elements in $\mathcal{G}$ are given by:*
  - $\sqcap\mathcal{F} = (\bigcap_{i\in[1,\ell]} {X_i}^\bullet, \bigcap_{i\in[1,\ell]} {X_i}^\times)$
  - $\sqcup\mathcal{F} = (\bigcup_{i\in[1,\ell]} {X_i}^\bullet, \bigcup_{i\in[1,\ell]} {X_i}^\times)$

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- $\mathcal{G}$ *is* **sup-generated**
- *But $\mathcal{G}$ is* **not complemented**

Edge-vertex correspondences: the building blocks

Definition

We define the operators $\delta^\bullet$, $\epsilon^\bullet$, $\epsilon^\times$, and $\delta^\times$ as follows:

	$\mathcal{G}^\times \rightarrow \mathcal{G}^\bullet$	$\mathcal{G}^\bullet \rightarrow \mathcal{G}^\times$

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where $\mathcal{G}_{X^\times}$ (resp. $\mathcal{G}_{X^\bullet}$) is the set of graphs with edge-set $X^\times$ (resp. vertex-set $X^\bullet$)

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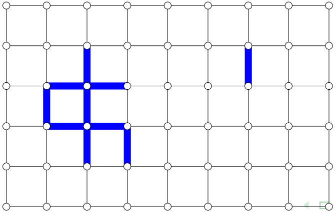

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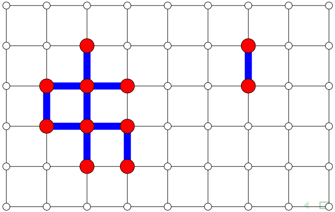

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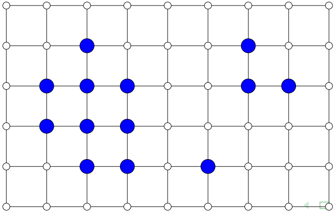

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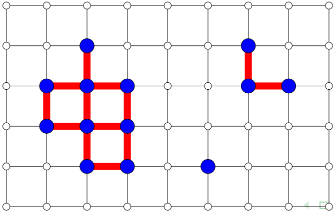

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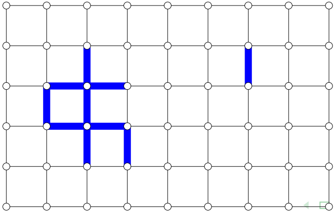

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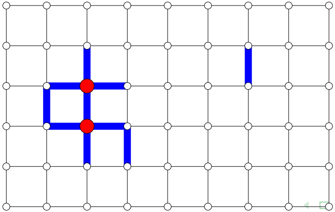

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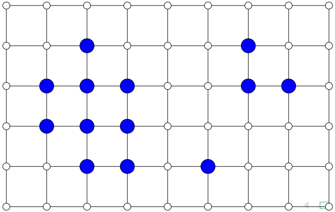

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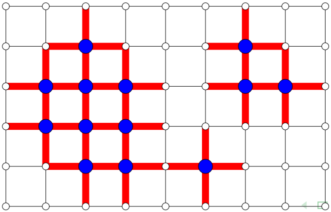

Adjunctions: reminder

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- Let $(\mathcal{L}_1, \leq_1)$ and $(\mathcal{L}_2, \leq_2)$ be two lattices
- Two operators $\epsilon : \mathcal{L}_1 \rightarrow \mathcal{L}_2$ and $\delta : \mathcal{L}_2 \rightarrow \mathcal{L}_1$ form an *adjunction* (ϵ, δ) if:
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- **If (ϵ, δ) is an adjunction, then ϵ is an erosion and δ is a dilation**:
  - ϵ preserves the infimum
  - δ preserves the supremum

Edge-vertex adjunctions

Property

1. *Both* $(\epsilon^{\times}, \delta^{\bullet})$ *and* $(\epsilon^{\bullet}, \delta^{\times})$ *are adjunctions*

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Important idea

- *To obtain operators acting on the lattices* $\mathcal{G}^{\bullet}$, $\mathcal{G}^{\times}$ *and* $\mathcal{G}$, *we will compose the operators of these basic adjunctions*

Vertex-dilation, vertex-erosion

Definition

- *We define* δ *and* ϵ *that act on* $\mathcal{G}^\bullet$ (i.e., $\mathcal{G}^\bullet \rightarrow \mathcal{G}^\bullet$) *by:*
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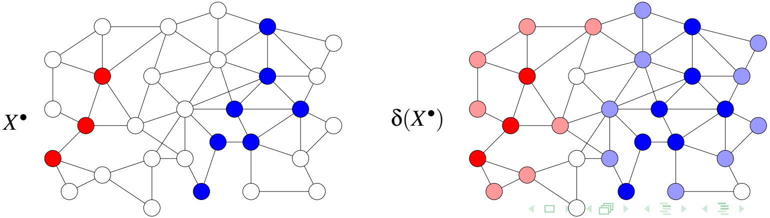
Property

- *The pair (ϵ, δ) is an adjunction*
- *They correspond exactly to the operators defined by Vincent*



Edge-dilation, edge-erosion

Definition (edge-dilation, edge-erosion)

- *We define* Δ *and* $\mathcal{E}$ *that act on* $\mathcal{G}^{\times}$ *by:*
  - $\Delta = \delta^{\times} \circ \delta^{\bullet}$ *and* $\mathcal{E} = \epsilon^{\times} \circ \epsilon^{\bullet}$

Property

- *The pair* $(\mathcal{E}, \Delta)$ *is an adjunction*

Graph-dilation, graph-erosion

Definition

- *We define, for any $X \in \mathcal{G}$, the operators $\delta \circledvee \Delta$ and $\epsilon \circledvee \mathcal{E}$ by:*
  - $\delta \circledvee \Delta(X) = (\delta(X^{\bullet}), \Delta(X^{\times}))$ *and*
  - $\epsilon \circledvee \mathcal{E}(X) = (\epsilon(X^{\bullet}), \mathcal{E}(X^{\times}))$

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Theorem

- *The lattice $\mathcal{G}$ is closed under the operators $\delta \,\textcircled{\scriptsize$\vee$}\, \Delta$ and $\epsilon \,\textcircled{\scriptsize$\vee$}\, \mathcal{E}$*

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- *The lattice $\mathcal{G}$ is closed under the operators $\delta \varovee \Delta$ and $\epsilon \varovee \mathcal{E}$*
- *The pair $(\epsilon \varovee \mathcal{E}, \delta \varovee \Delta)$ is an adjunction*

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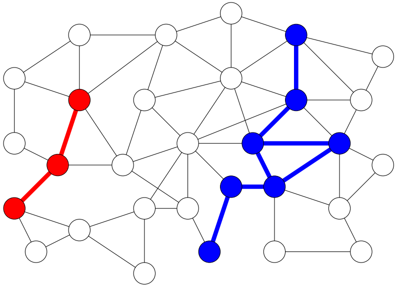
- *We define, for any $X \in \mathcal{G}$, the operators $\delta \varovee \Delta$ and $\epsilon \varovee \mathcal{E}$ by:*
  - $\delta \varovee \Delta(X) = (\delta(X^\bullet), \Delta(X^\times))$ *and*
  - $\epsilon \varovee \mathcal{E}(X) = (\epsilon(X^\bullet), \mathcal{E}(X^\times))$

Theorem

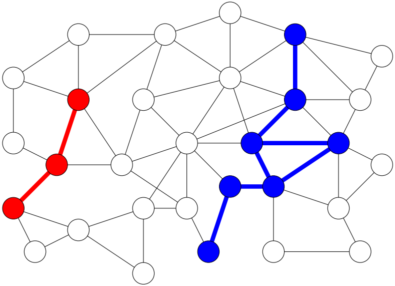
- *The lattice $\mathcal{G}$ is closed under the operators $\delta \varovee \Delta$ and $\epsilon \varovee \mathcal{E}$*
- *The pair $(\epsilon \varovee \mathcal{E}, \delta \varovee \Delta)$ is an adjunction*
- *The operators $\delta \varovee \Delta$ and $\epsilon \varovee \mathcal{E}$ are respectively a dilation and an erosion acting on the lattice $(\mathcal{G}, \sqsubseteq)$*

Graph-dilation: example

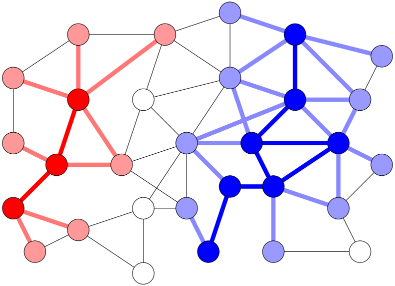


X (red & blue)

Graph-dilation: example



X (red & blue)



$\delta \ovee \Delta(X)$ (red & blue)

Filters: reminder

- A *filter* is an operator α acting on a lattice $\mathcal{L}$, which is
  1. increasing: $\forall X, Y \in \mathcal{L}$, $\alpha(X) \leq \alpha(Y)$ whenever $X \leq Y$; and
  2. idempotent: $\forall X \in \mathcal{L}$, $\alpha(\alpha(X)) = \alpha(X)$

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- **Composing the two operators of an adjunction yields an opening or a closing depending on the composition order**

Openings, closings: the classical ones

Definition

We define

1. γ_1 *and* ϕ_1, $\mathcal{G}^\bullet \to \mathcal{G}^\bullet$, *by* $\gamma_1 = \delta \circ \epsilon$ *and* $\phi_1 = \epsilon \circ \delta$

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3. $\gamma \varovee \Gamma_1$ *and* $\phi \varovee \Phi_1$ *by respectively* $\gamma \varovee \Gamma_1(X) = (\gamma_1(X^\bullet), \Gamma_1(X^\times))$ *and* $\phi \varovee \Phi_1(X) = (\phi_1(X^\bullet), \Phi_1(X^\times))$, *for any graph* X *in* $\mathcal{G}$

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Theorem (graph-openings, graph-closings)

- *The operators* γ_1 *and* Γ_1 *are openings. The operators* Φ_1 *and* ϕ_1 *are closings*
- *The operator* $\gamma \oslash \Gamma_1$ *is an opening. The operator* $\phi \oslash \Phi_1$ *is a closing*

Openings, closings: the half ones

Definition

We define

1. $\gamma_{1/2}$ *and* $\phi_{1/2}$, $\mathcal{G}^\bullet \to \mathcal{G}^\bullet$, *by* $\gamma_{1/2} = \delta^\bullet \circ \epsilon^\times$ *and* $\phi_{1/2} = \epsilon^\bullet \circ \delta^\times$

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3. $\gamma \circledvee \Gamma_{1/2}$ *and* $\phi \circledvee \Phi_{1/2}$ *by* $\gamma \circledvee \Gamma_{1/2}(X) = (\gamma_{1/2}(X^\bullet), \Gamma_{1/2}(X^\times))$ *and* $\phi \circledvee \Phi_{1/2}(X) = (\phi_{1/2}(X^\bullet), \Phi_{1/2}(X^\times))$, *for any graph* X *in* $\mathcal{G}$

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3. $\gamma \varovee \Gamma_{1/2}$ *and* $\phi \varovee \Phi_{1/2}$ *by* $\gamma \varovee \Gamma_{1/2}(X) = (\gamma_{1/2}(X^\bullet), \Gamma_{1/2}(X^\times))$ *and* $\phi \varovee \Phi_{1/2}(X) = (\phi_{1/2}(X^\bullet), \Phi_{1/2}(X^\times))$, *for any graph* X *in* $\mathcal{G}$

Theorem (half-openings, half-closings)

- *The operators* $\gamma_{1/2}$ *and* $\Gamma_{1/2}$ *are openings. The operators* $\phi_{1/2}$ *and* $\Phi_{1/2}$ *are closings*
- *The operator* $\gamma \varovee \Gamma_{1/2}$ *is an opening. The operator* $\phi \varovee \Phi_{1/2}$ *is a closing.*

Illustration: openings

A graph X (red)

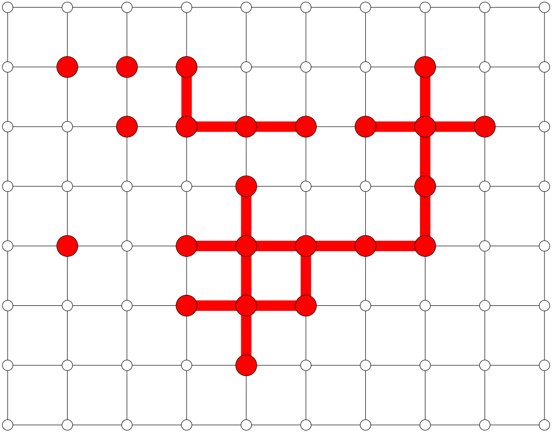


Illustration: openings

A graph X (red, blue)
$\gamma \circledcirc \Gamma_{1/2}(X)$ (red)

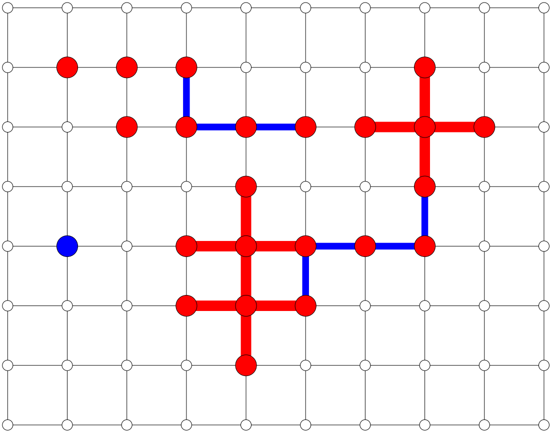
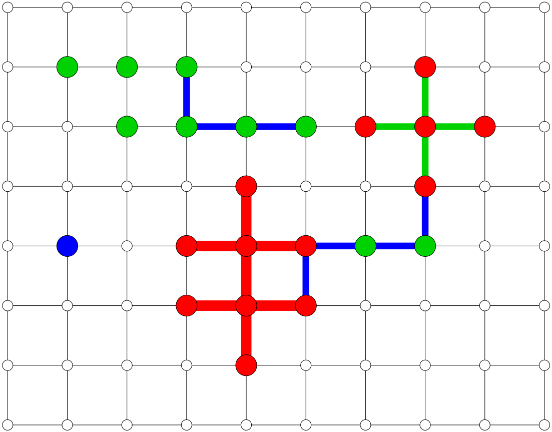


Illustration: openings

A graph X (red, blue, green)
$\gamma \circledvee \Gamma_{1/2}(X)$ (red, green)
$\gamma \circledvee \Gamma_1(X)$ (red)



Building hiearchies

Definition

Let $\lambda \in \mathbb{N}$. Let i and j be the quotient and the remainder in the integer division of λ by 2.

- *We set:*
  - $\gamma \circledvee \Gamma_{\lambda/2} = (\delta \circledvee \Delta)^i \circ (\gamma \circledvee \Gamma_{1/2})^j \circ (\epsilon \circledvee \mathcal{E})^i$
  - $\phi \circledvee \Phi_{\lambda/2} = (\epsilon \circledvee \mathcal{E})^i \circ (\phi \circledvee \Phi_{1/2})^j \circ (\delta \circledvee \Delta)^i$

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Property

- *The families $\{\gamma \circledvee \Gamma_{\lambda/2} \mid \lambda \in \mathbb{N}\}$ and $\{\phi \circledvee \Phi_{\lambda/2} \mid \lambda \in \mathbb{N}\}$ are granulometries:*
  - $\forall \lambda \in \mathbb{N}$, $\gamma \circledvee \Gamma_{\lambda/2}$ *is an opening on* $\mathcal{G}$ *and* $\phi \circledvee \Phi_{\lambda/2}$ *is a closing on* $\mathcal{G}$
  - $\forall \mu, \nu \in \mathbb{N}$ *and* $\forall X \in \mathcal{G}$, $\mu \leq \nu$ *implies* $\gamma \circledvee \Gamma_{\nu/2}(X) \sqsubseteq \gamma \circledvee \Gamma_{\mu/2}(X)$ *and* $\phi \circledvee \Phi_{\mu/2}(X) \sqsubseteq \phi \circledvee \Phi_{\nu/2}(X)$

Iterated filters

Definition

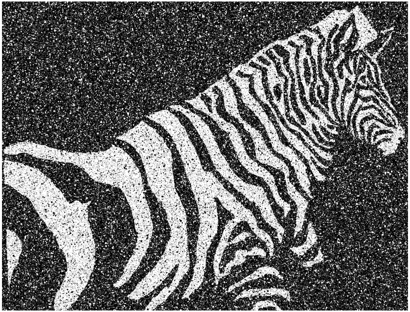
- *We define* $ASF_{\lambda/2}$
  - *by the identity on graphs when* $\lambda = 0$
  - *by* $ASF_{\lambda/2} = \gamma \circledvee \Gamma_{\lambda/2} \circ \phi \circledvee \Phi_{\lambda/2} \circ ASF_{(\lambda-1)/2}$ *otherwise*

Property

- *The family* $\{ASF_{\lambda/2} \mid \lambda \in \mathbb{N}\}$ *is a family of alternate sequential filters:*
  - $\forall \mu, \nu \in \mathbb{N}$, $\mu \geq \nu$ *implies* $ASF_{\mu/2} \circ ASF_{\nu/2} = ASF_{\mu/2}$

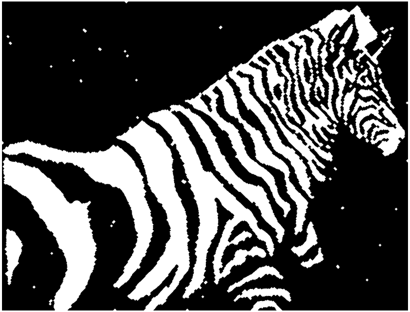
ASF: illustration for binary image filtering

Original



Noisy

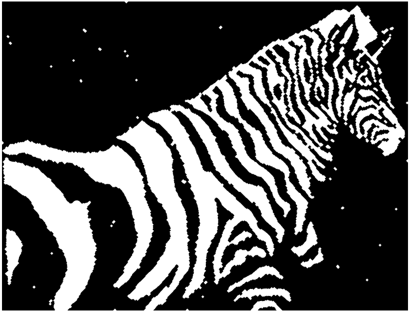
ASF: illustration for binary image filtering



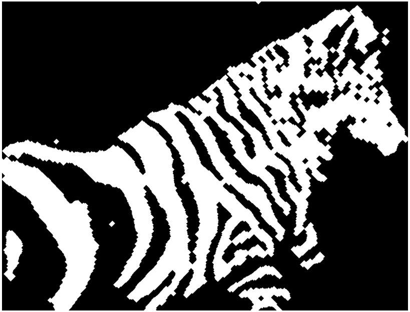
graph ASF

Classical ASF

ASF: illustration for binary image filtering

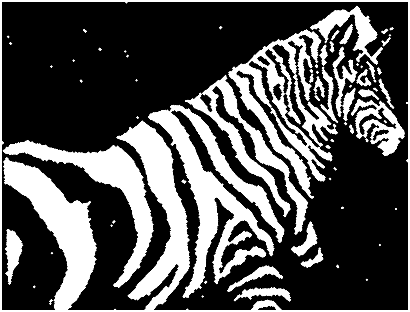


graph ASF



Classical ASF of double size

ASF: illustration for binary image filtering



graph ASF

Classical ASF (double resolution)

Other adjunctions on graphs

1. (α_1, β_1) such that $\forall X \in \mathcal{G}$, $\alpha_1(X) = (\mathbb{G}^\bullet, X^\times)$ and $\beta_1(X) = (\delta^\bullet(X^\times), X^\times)$
2. (α_2, β_2) such that $\forall X \in \mathcal{G}$, $\alpha_2(X) = (X^\bullet, \epsilon^\times(X^\bullet))$ and $\beta_2(X) = (X^\bullet, \emptyset)$
   - **α_1 and α_2 are both a closing and an erosion;**
     β_1 and β_2 are both an opening and dilation
3. (α_3, β_3) such that $\forall X \in \mathcal{G}$, $\alpha_3(X) = (\epsilon^\bullet(X^\times), \epsilon^\times \circ \epsilon^\bullet(X^\times))$ and $\beta_3(X) = (\delta^\bullet \circ \delta^\times(X^\bullet), \delta^\times(X^\bullet))$
   - **α_3 depends only on the edge-set;**
     β_3 only on the vertex-set

Perspectives

- Extension to node and edge-**weighted graphs**

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- Study of **levellings** in this framework

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- Study of **levellings** in this framework
- Extension to **simplicial/cubical complexes**
- Relation with **connexion** and **hyper-connexion**
- Embedding of the vertices in a **metric space**