Master 2 "SIS"

# Digital Geometry

## Topic 1 :
## Introduction: digital images and discretization models

**Yukiko Kenmochi**

October 3, 2012

# What is digital geometry?

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Real objects are represented in a discrete way as finite sets because computers can handle only finite structures.

*(Ex. images obtained by a CCD camera or computer tomography, CT or MRI images, satellite images, etc.)*

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- exact computation (no computation error)
- precision analysis due to image resolution
- a finite number of local discrete shapes
- efficient algorithms
- ...

# Related domains to digital geometry

- discrete geometry and topology
- graph theory
- computational geometry
- combinatorial geometry
- number theory
- approximation and estimation
- mathematical morphology
- image processing and analysis
- computer graphics
- computer vision
- pattern recognition
- ...

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## in terms of digital image "dimension"

- 1980s: work for 2D images
- 1990s: work for 3D images
- 2000s: work for $n$D images

# Lecture schedule

| | | |
|---|---|---|
| 3/10/2012 | **Topic 1:** | Introduction: digital images and discretization models |
| 10/10/2012 | **Topic 2:** | Discrete objects and their boundaries: adjacency graph representation |
| 17/10/2012 | **Topic 3:** | Discrete surfaces and object boundaries: from a grid point set to a polygonal mesh |
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- **Assignments:** Three mini projects are planed.
- **Grading policy:** 50 % (assignments) + 50 % (examination)

# Continuous and digital images

**Definition (Continuous image)**

*An nD image is defined as a function* $\mathcal{I} : \mathbb{R}^n \rightarrow V$ *where* $V$ *is a value space containing at least two elements.*

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**Definition (Digital image)**

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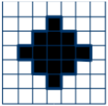
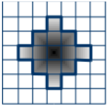
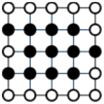
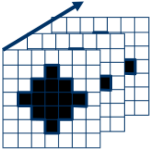
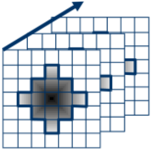
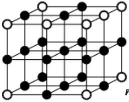
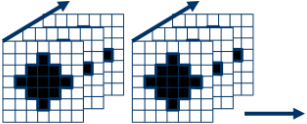
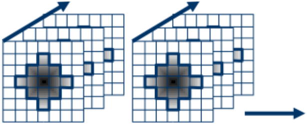
Examples for $V$:

- scalars: $\mathbb{R}$, $\mathbb{Z}$, $\{0, 1\}$, $\{x \in \mathbb{Z} : 0 \leq x \leq 255\}$, ...
- vectors: $\mathbb{R}^k$, $\mathbb{Z}^k$, ...
- tensors

# Digital images and discrete spaces

| dim. | binary image | gray image | discrete space |
|---|---|---|---|
| 2D |  |  |  $\mathbb{Z}^2$ |
| 3D |  |  |  $\mathbb{Z}^3$ |
| 4D |  |  | |

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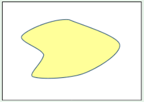
- **discretization:** *sample the value of an analog signal at regular intervals;*
- **quantization:** *round those samples to a fixed set of numbers such as integers.*

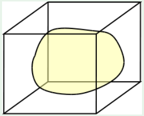
For an $n$D digital image, those **regular intervals** make a grid, *i.e.* the **discrete space** $\mathbb{Z}^n$.

# Continuous space and discrete space

## object in continuous space

$S \subset \mathbb{R}^n$

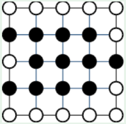


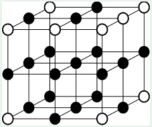


↓ **Discretization**

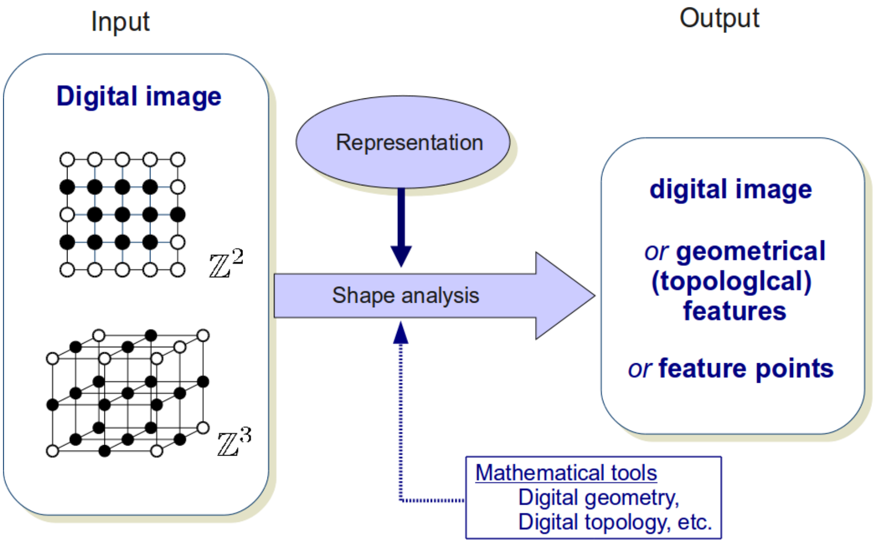
## object in discrete space

$D(S) \subset \mathbb{Z}^n$





# Discrete shape analysis



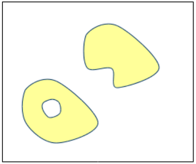
# Digital geometry and topology

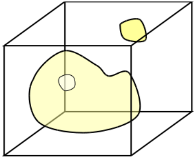
## Geometrical features of discrete objects

**Examples:** straightness, planarity (linearity), circularity, sphericity (roundness), convexity, concavity, curvature, perimeter (length), area, volume, centroid (moments), ...

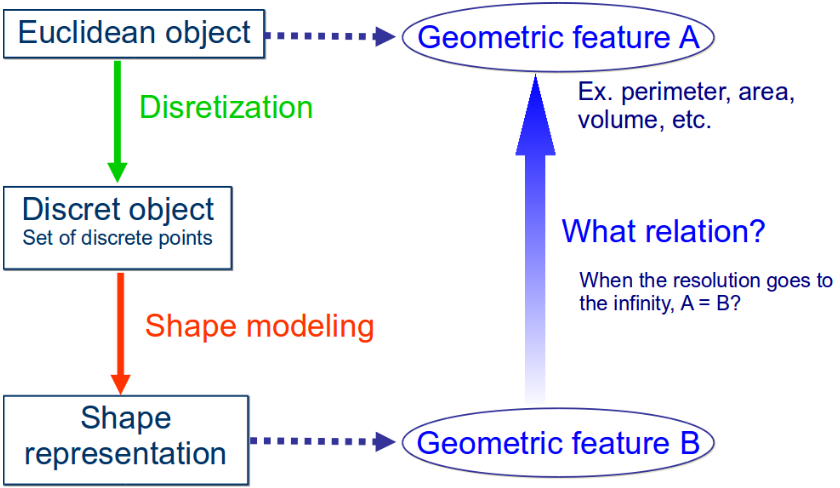
## Topological features of discrete objects

**Examples:** object boundary, curve, surface, number of objects (of connected components), number of holes in an object, shape deformation preserving the topology, ....





# Discretization and shape analysis

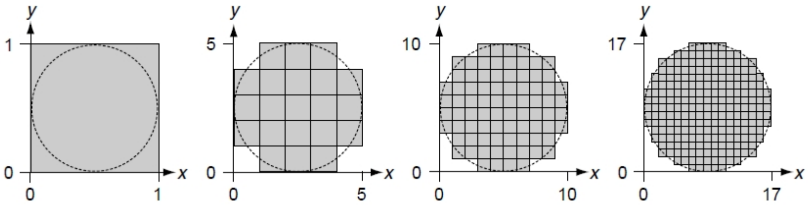


# Image resolution and discrete space

Image (grid) resolution is the inverse of the grid interval, which is generally set to be 1.

## Definition (Multigrid discrete space)

*Let $h > 0$ be a grid resolution and $\mathbb{Z}_h = \{i/h : i \in \mathbb{Z}\}$; then $\mathbb{Z}_h^n$ is the set of nD discrete points in a grid of resolution of $h$.*




(Klette and Rosenfeld, 2003)

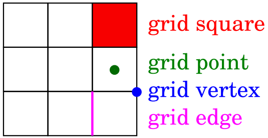
# Grid points and grid cells

## Definition (Grid points)

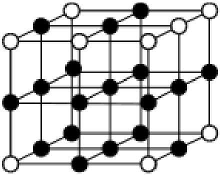
*The grid point set is* $\mathbb{Z}^n$ $(\mathbb{Z}_h^n)$.

## Definition (Grid cells)

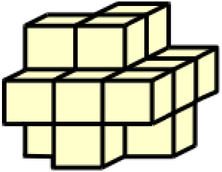
*A grid vertex is also called a 0-cell; a grid edge is a 1-cell; a grid square is a 2-cell; a grid cube is called a 3-cell;.....*




2D grid



3D grid point model



3D grid cell model

# Properties expected to be preserved by discretization

### symmetry

If **S** is symmetric, its discretization $D(\mathbf{S})$ is symmetric in the same way.

### connectedness

If **S** is connected, $D(\mathbf{S})$ is connected as well.

### dimension

If **S** is a curve (surface), $D(\mathbf{S})$ is a curve (surface) as well.

### topology

**S** and $D(\mathbf{S})$ are topologically equivalent (*i.e.*, homeomorphic).

### multigrid convergence

When the image resolution $h$ goes to the infinity, the series of $D_h(\mathbf{S})$ converges such that their limit is equal to **S**.

# Discretization models

Let $S \subset \mathbb{R}^n$ be an original object.

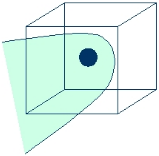
**Definition (Gauss discretization)**

*The* **Gauss discretization** $G(\mathbf{S})$ *is the union of the n-cells with center points in S.*

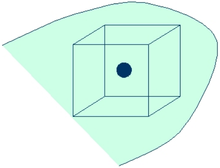
**Definition (Jordan discretization)**

*The* **inner Jordan discretization** $J^{-}(\mathbf{S})$ *is the union of n-cells that are completely contained in S. The* **outer Jordan discretization** $J^{+}(\mathbf{S})$ *is the union of all such n-cells that have nonempty intersections with S.*

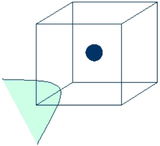
The outer Jordan discretization is also called **super-cover discretization**.



$G(\mathbf{S})$

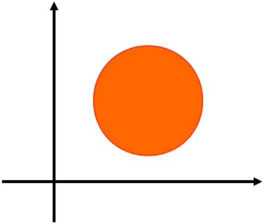


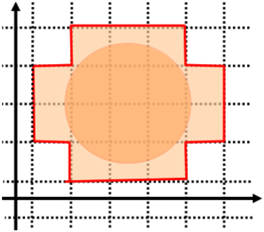
$J^{-}(\mathbf{S})$



$J^{+}(\mathbf{S})$

# Example of super-cover discretization: disk





For $\mathbf{S} = \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x} - \mathbf{c}\| \leq r\}$,

$$D(\mathbf{S}) = \{\mathbf{p} \in \mathbb{Z}^2 : Cell(\mathbf{p}) \cap \mathbf{S} \neq \emptyset\}$$

where $Cell(\mathbf{p}) = \left[p_x - \frac{1}{2}, p_x + \frac{1}{2}\right] \times \left[p_y - \frac{1}{2}, p_y + \frac{1}{2}\right]$.

(Andrès, 2008)

# Properties of super-cover discretization

- $D(\mathbf{X} \cup \mathbf{Y}) = D(\mathbf{X}) \cup D(\mathbf{Y})$,
- $D(\mathbf{X}) = \cup_{\mathbf{p} \in \mathbf{X}} D(\{\mathbf{p}\})$,
- $D(\mathbf{X} \cap \mathbf{Y}) \subseteq D(\mathbf{X}) \cap D(\mathbf{Y})$,
- if $\mathbf{X} \subseteq \mathbf{Y}$, then $D(\mathbf{X}) \subseteq D(\mathbf{Y})$.

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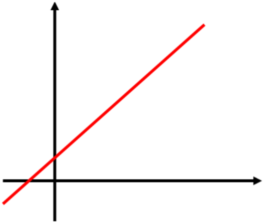
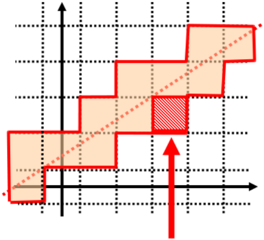
# Example of super-cover discretization: straight line






**Curve structure (topology) is not always preserved!**

## Solution

**Minimal-cover**

**Symmetry is not always preserved.**

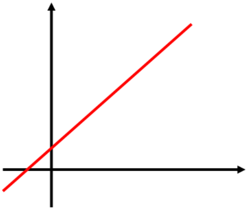
# Grid-intersection discretization

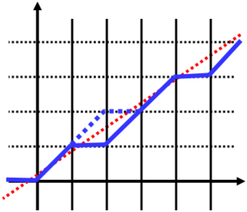
Gauss and inner Jordan discretization are not appropriate for curves or arcs.

## Definition (Grid-intersection discretization)

*Let $\gamma \subset \mathbb{R}^2$ be a curve. The* **grid-intersection discretization** $R(\gamma)$ *is the set of all grid points that are closest (in Euclidean distance) to the intersections points of $\gamma$ with the grid lines.*

This can be extended to 3D curves or arcs.





When two grid points are in the same distance from the intersection, the choice is needed.

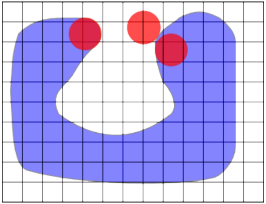
Thus, symmetry is not always preserved.

# Topology preservation and discretization

## Definition (Compatibility (Pavlidis, 1982))

*A closed set $S \in \mathbb{R}^2$ and a grid whose square cells have diameter $h$ are* **compatible** *if:*

1. *there exists a number $d > h$ such that for each boundary point $\mathbf{x}$ of each connected component $S'$ of $S$, there is a closed ball with diameter $d$ that is tangent to the boundary of $S'$ at $\mathbf{x}$ and lies entirely within $S'$;*
2. *the same is also true for the closure of the complement of $S$.*



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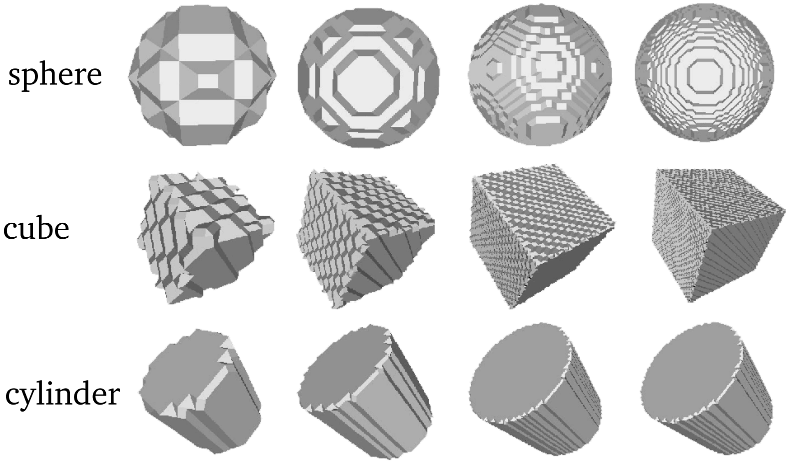
2. *the same is also true for the closure of the complement of $S$.*

This is true only for **Gauss discretization**.

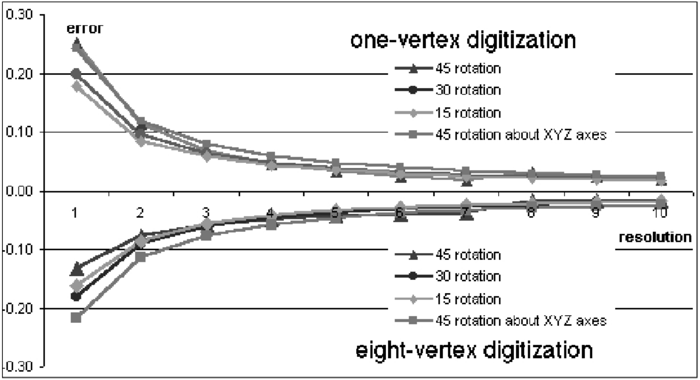
This can be **generalized** by doubling the diameter of the tangent ball (Latecki, 1998).

# Multigrid discrete objects



# Volume computation and multigrid convergence

Volumes for a cube with different viewpoints are computed in different grid resolutions.




Local volume elements converge towards the true volume value (known since end of 19th century: C. Jordan, G. Peano et al.).

# Surface area computation and multigrid convergence

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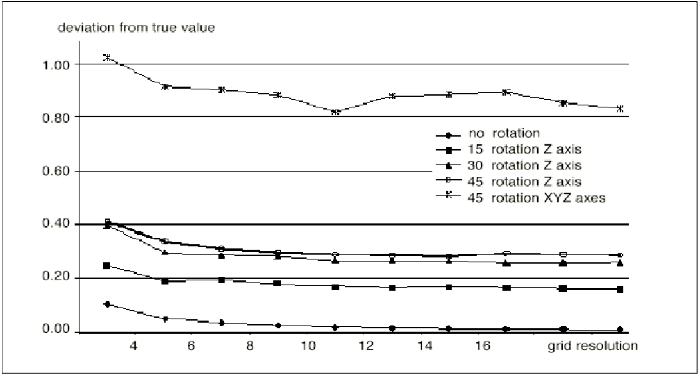
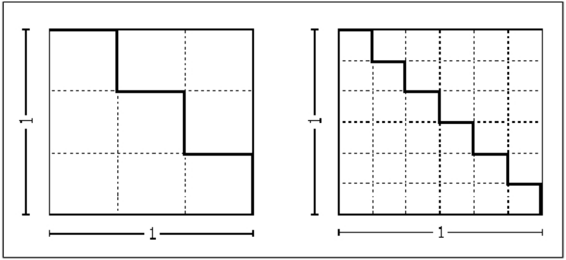
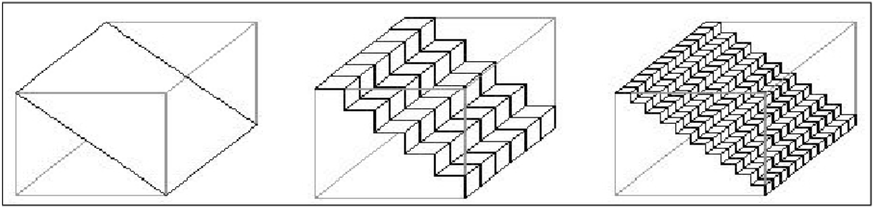



## Problem

**Different** surface area estimates for one cube from different perspectives (using marching cubes algorithms).

# Multigrid convergence for perimeter and surface area



# How to obtain correct geometrical features from discrete objects?

Principal idea is to add some **hypothesis on the shape**: for example,

- dimension of object,
- number of objects,
- number of holes,
- smoothness (of curves and surfaces),
- linearity,
- circularity,
- ...

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**You need to know what you would like to see.**

# References


- Reinhard Klette and Azriel Rosenfeld.
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- David Coeurjolly, Annick Montanvert, et Jean-Marc Chassery.
  "Géométrie discrète et images numériques", Lavoisier, 2007.

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