Bounding the size of an almost-equidistant set in Euclidean space

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Abstract

A set of points in d-dimensional Euclidean space is *almost equidistant* if among any three points of the set, some two are at distance 1. We show that an almost-equidistant set in $\mathbb{R}^d$ has cardinality $O(d^{4/3})$.

1 Introduction

A set of lines through the origin of Euclidean d-space $\mathbb{R}^d$ is *almost orthogonal* if among any three of the lines, some two are orthogonal. Erdős asked (see [12]) what is the largest cardinality of an almost-orthogonal set of lines in $\mathbb{R}^d$? By taking the union of two sets of d pairwise orthogonal lines, we see that $2d$ is possible. Rosenfeld [12] showed that $2d$ is the maximum by considering the eigenvalues of the Gramian of the unit vectors spanning the lines. His result was subsequently given simpler proofs by Pudlák [11] and Deaett [6].

In this note we consider the analogous notion obtained by replacing orthogonal pairs of lines by pairs of points at unit distance. A subset V of Euclidean d-space $\mathbb{R}^d$ is *almost equidistant* if among any three points in V, some two are at Euclidean distance 1. We investigate the largest size, which we denote by $f(d)$, of an almost-equidistant set in $\mathbb{R}^d$. Although asking for the size of this function is a very natural question, it seems to be harder than the question of Erdős, which can be refomulated as asking for the largest size of an almost-equidistant set on a sphere of radius $1/\sqrt{2}$ in $\mathbb{R}^d$. Before stating our main result, we give an overview of what is known about $f(d)$.

Bezdek, Naszódi and Visy [5] showed that $f(2) \leq 7$, and István Talata (personal communication, 2007) showed that the only almost-equidistant set in $\mathbb{R}^2$ with 7 points is the Moser spindle. Györey [7] showed that $f(3) \leq 10$ and that there is a unique almost-equidistant set of 10 points in $\mathbb{R}^3$, a configuration originally considered by Nechushtan [9]. The Moser spindle can be generalized to higher dimensions, giving an almost-equidistant set of $2d + 3$ points in $\mathbb{R}^d$ [4]. (We mention that Bezdek and Langi [4] considered the variant of Erdős's problem where the radius of the sphere is arbitrary instead of $1/\sqrt{2}$.) A

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construction of Larman and Rogers [8] shows that $f(5) \geq 16$. Since there does not exist a set of $d+2$ points in $\mathbb{R}^d$ that are pairwise at distance 1, it follows that $f(d) \leq R(d+2,3)-1$, where the Ramsey number $R(a,b)$ is the smallest n such that whenever each edge of the complete graph on n vertices is coloured blue or red, there is either a blue clique of size a or a red clique of size b. Ajtai, Komlós, and Szemerédi [1] showed $R(k,3) = O(k^2/\log k)$, which implies the asymptotic upper bound $f(d) \leq O(d^2/\log d)$. Balko, Pór, Scheuer, Swanepoel, and Valtr [3] generalized the Nechushtan configuration to higher dimensions, giving $f(d) \geq 2d+4$ for all $d \geq 3$. They also obtained the asymptotic upper bound $f(d) = O(d^{3/2})$ by an argument based on Deaett's paper [6]. Using computer search and ad hoc geometric arguments, they obtained the following bounds for small d: $f(4) \leq 13$, $f(5) \leq 20$, $18 \leq f(6) \leq 26$, $20 \leq f(7) \leq 34$, and $f(9) \geq f(8) \geq 24$. Polyanskii [10] subsequently improved the asymptotic upper bound to $f(d) = O(d^{13/9})$.

In this note we obtain a further improvement to the upper bound.

Theorem 1. *An almost-equidistant set in $\mathbb{R}^d$ has cardinality $O(d^{4/3})$.*

Its proof is based on the approach of [3] to show the upper bound $O(d^{3/2})$, which is in turn based on Deaett's proof [6] of Rosenfeld's result. Before proving Theorem 1 in Section 3, we establish notation and collect some lemmas in the next section.

2 Preliminaries

We denote the Euclidean norm of $x \in \mathbb{R}^d$ by $\|x\|$ and the inner product of $x, y \in \mathbb{R}^d$ by $\langle x, y \rangle$. The cardinality of a finite set A is denoted by $|A|$. We call a finite non-empty subset C of $\mathbb{R}^d$ (the vertex set of) a *unit simplex* if the distance between any two points in C equals 1. It has already been mentioned in the Introduction that if C is a unit simplex then $|C| \leq d+1$. Given any finite $V \subset \mathbb{R}^d$, we define the *unit-distance graph* $G = (V, E)$ *on* V to be the graph with $vw \in E$ iff $\|v - w\| = 1$. Thus, $C \subset V$ is a unit simplex iff it is a clique in G. We denote the set of neighbours of $v \in V$ in G by $N(v)$.

The following well-known lemma gives a lower bound for the rank of a square matrix in terms of its entries [2, 6, 11].

Lemma 1. *For any non-zero $n \times n$ symmetric matrix $A = [a_{i,j}]$,*

$$\operatorname{rank}(A) \geq \frac{(\sum_i a_{i,i})^2}{\sum_{i,j} a_{i,j}^2}.$$

For the sake of completeness, we include the proofs of the following three lemmas on the vertices and centroids of unit simplices.

Lemma 2. *Let C be a unit simplex with centroid $c = \frac{1}{|C|}\sum_{v \in C} v$. Then*

$$\|v - c\|^2 = \frac{1}{2}\left(1 - \frac{1}{|C|}\right) \quad \text{for all } v \in C,$$

and

$$\left\langle v - c, v' - c \right\rangle = -\frac{1}{2\,|C|} \quad \text{for all distinct } v, v' \in C.$$

Proof. We may translate C so that c is the origin o. Write $C = \{p_1, \dots, p_k\}$. By symmetry, $\alpha := \|p_i\|^2$ is independent of i, and $\beta := \langle v_i, v_j \rangle$ $(i \neq j)$ is independent of i and j. Then

$$0 = \left\| \sum_{i=1}^{k} p_i \right\|^2 = k\alpha + k(k-1)\beta$$

and

$$1 = \|p_i - p_j\|^2 = 2\alpha - 2\beta.$$

Solving these two linear equations in α and β, we obtain $\alpha = \frac{1}{2} - \frac{1}{2k}$ and $\beta = -\frac{1}{2k}$. □

Lemma 3. *Let C be a unit simplex with centroid $c = \frac{1}{|C|}\sum_{v \in C} v$, and let $F \subset C$ be a unit simplex with centroid $f = \frac{1}{|F|}\sum_{v \in F} v$. Then*

$$\|c - f\|^2 = \frac{1}{2}\left(\frac{1}{|F|} - \frac{1}{|C|}\right).$$

Proof. Let $k := |C|$, $\ell := |F|$. Then

$$\begin{aligned} \|f - c\|^2 &= \left\| \frac{1}{\ell} \sum_{v \in F} (v - c) \right\|^2 = \frac{1}{\ell^2}\left(\ell \cdot \frac{1}{2}\Big(1 - \frac{1}{k}\Big) - \frac{\ell(\ell-1)}{2k} \right) \quad \text{by Lemma 2} \\ &= \frac{1}{2}\left(\frac{1}{\ell} - \frac{1}{k}\right). \end{aligned}$$

□

Lemma 4. *Let A and B be disjoint unit simplices with centroids $a = \frac{1}{|A|}\sum_{v \in A} v$ and $b = \frac{1}{|B|}\sum_{v \in B} v$, respectively, such that $A \cup B$ is also a unit simplex. Then*

$$\|a - b\|^2 = \frac{1}{2}\left(\frac{1}{|A|} + \frac{1}{|B|}\right).$$

Proof. Let $C := A \cup B$ have centroid c. Then by Lemma 3, $\|a - c\|^2 = \frac{1}{2}\left(\frac{1}{|A|} - \frac{1}{|C|}\right)$ and $\|b - c\|^2 = \frac{1}{2}\left(\frac{1}{|B|} - \frac{1}{|C|}\right)$. It follows that

$$\begin{aligned} \|a - b\|^2 &= (\|a - c\| + \|b - c\|)^2 \\ &= \frac{1}{2}\left(\frac{1}{|A|} + \frac{1}{|B|} - \frac{2}{|C|}\right) + \sqrt{\left(\frac{1}{|A|} - \frac{1}{|C|}\right)\left(\frac{1}{|B|} - \frac{1}{|C|}\right)} \\ &= \frac{1}{2}\left(\frac{1}{|A|} + \frac{1}{|B|}\right) \end{aligned}$$

□

3 Proof of Theorem 1

Let G be the unit-distance graph of a given almost-equidistant set V. Then the complement of G is K_3-free, and the non-neighbours of any vertex form a unit simplex. Let C be a clique of maximum cardinality in G. Write $k = |C|$. Each $v \in V \setminus C$ is a non-neighbour of some point in C, and it follows that $|V| \leq |C| + |C|\,k = k^2 + k$. Thus, without loss of generality, $k > d^{2/3}$.

We split V up into two parts, each to be bounded separately. Let

$$N = \left\{ v \in V : |N(v) \cap C| \geq k - k^{4/3} d^{-2/3} \right\}.$$

Note that $k^{4/3}d^{-2/3} = O(d^{-1/3})k$. We first bound the complement of N. Consider the set

$$X = \{(u, v) \in C \times V \setminus N : uv \notin E(G)\}.$$

For each $v \in V \setminus N$, there are more than $k^{4/3}d^{-2/3}$ points $u \in C$ such that $u \notin N(v)$, hence $|X| > k^{4/3}d^{-2/3} |V \setminus N|$. On the other hand, for each $u \in C$, the set of non-neighbours of u forms a clique, so has cardinality at most k, and $|X| \leq |C| k = k^2$. It follows that

$$|V \setminus N| < k^{2/3} d^{2/3}. \tag{1}$$

Next, we estimate $|N|$. Without loss of generality, $\frac{1}{k}\sum_{v \in C} v = o$ and $N = \{v_1, \ldots, v_n\}$. We want to apply Lemma 1 to the $n \times n$ matrix $A = [\langle v_i, v_j \rangle]$, which has rank at most d.

Claim 1. *For each $i = 1 \ldots, n$, $\|v_i\|^2 = \frac{1}{2} + O(k^{-1/3}d^{-1/3})$, and for each $v_iv_j \in E(G)$, $\langle v_i, v_j \rangle = O(k^{-1/3}d^{-1/3})$.*

Proof of Claim 1. Let $C_i := N(v_i) \cap C$, $k_i := |C_i|$, and $c_i := \frac{1}{k_i}\sum_{v \in C_i} v$. Then $k_i \geq k - k^{4/3}d^{-2/3}$. By Lemma 4 applied to $A = \{v_i\}$ and $B = C_i$, $\|v_i - c_i\|^2 = \frac{1}{2}\left(1 + \frac{1}{k_i}\right)$, hence $\|v_i - c_i\| = \frac{1}{\sqrt{2}} + O(k^{-1})$. By Lemma 3 applied to C and $F = C_i$,

$$\|c_i\| = \sqrt{\frac{1}{2}\left(\frac{1}{k_i} - \frac{1}{k}\right)} \leq \sqrt{\frac{1}{2}\left(\frac{1}{k - k^{4/3}d^{-2/3}} - \frac{1}{k}\right)} = O(k^{-1/3}d^{-1/3}).$$

By the triangle inequality,

$$\|v_i\| = \|v_i - c_i\| + O(\|c_i\|) = \frac{1}{\sqrt{2}} + O(k^{-1/3}d^{-1/3}),$$

and $\|v_i\|^2 = \frac{1}{2} + O(k^{-1/3}d^{-1/3})$. Also, $2\langle v_i, v_j \rangle = \|v_i\|^2 + \|v_j\|^2 - 1 = O(k^{-1/3}d^{-1/3})$. □

Claim 2. *For each $i = 1, \ldots, n$,*

$$\sum_{\substack{j=1 \\ v_iv_j \notin E(G)}}^{n} \langle v_i, v_j \rangle^2 = O(k^{2/3}d^{-1/3}).$$

Proof of Claim 2. The non-neighbours $N \setminus N(v_i)$ of v_i form a unit simplex with cardinality $t := |N \setminus N(v_i)| \leq k$ and with centroid c, say. If $t = d + 1$, remove one point v_j from the unit simplex, which decreases the sum by $\langle v_i, v_j \rangle^2 = O(1)$. Thus, without loss of generality, $t \leq d$, and there exists a point $p \in \mathbb{R}^d$ such that $p - c$ is orthogonal to the affine hull of $N \setminus N(v_i)$, $\|p - c\| = 1/\sqrt{2t}$, the set $\{v_j - p : v_j \in N \setminus N(v_i)\}$ is orthogonal, and $\|v_j - p\| = 1/\sqrt{2}$ for each non-neighbour v_j of v_i. Then, by the finite Bessel inequality,

$$\sum_{v_j \in N \setminus N(v_i)} \langle v_i, v_j - p \rangle^2 \leq \frac{1}{2} \|v_i\|^2,$$

hence by applying Cauchy–Schwarz a few times,

$$
\begin{aligned}
\sum_{v_j \in N \setminus N(v_i)} \langle v_i, v_j \rangle^2 &= \sum_{v_j \in N \setminus N(v_i)} \big(\langle v_i, v_j - p \rangle + \langle v_i, p - c \rangle + \langle v_i, c \rangle \big)^2 \\
&\le 3 \sum_{v_j \in N \setminus N(v_i)} \big(\langle v_i, v_j - p \rangle^2 + \langle v_i, p - c \rangle^2 + \langle v_i, c \rangle^2 \big) \\
&\le 3 \left(\frac{1}{2} \|v_i\|^2 + t \|v_i\|^2 \|p - c\|^2 + t \|v_i\|^2 \|c\|^2 \right) \\
&\le 3 (\|v_i\|^2 + t \|v_i\|^2 \|c\|^2). \qquad (2)
\end{aligned}
$$

By Claim 1, $\|v_i\|^2 = \frac{1}{2} + O(k^{-1/3} d^{-1/3})$ and

$$
\begin{aligned}
\|c\|^2 = \left\| \frac{1}{t} \sum_{v_j \in N \setminus N(v_i)} v_j \right\|^2 &= \frac{1}{t^2} \Bigg(\sum_{v_j \in N \setminus N(v_i)} \|v_j\|^2 + \sum_{\substack{v_j, v_{j'} \in N \setminus N(v_i) \\ v_j \ne v_{j'}}} \langle v_j, v_{j'} \rangle \Bigg) \\
&\le \frac{1}{t^2} \left(t \left(\frac{1}{2} + O(k^{-1/3} d^{-1/3}) \right) + t(t-1) O(k^{-1/3} d^{-1/3}) \right) \\
&= \frac{1}{2t} + O(k^{-1/3} d^{-1/3}).
\end{aligned}
$$

Therefore,

$$
t \|c\|^2 = \frac{1}{2} + O(t k^{-1/3} d^{-1/3}) = O(k^{2/3} d^{-1/3}).
$$

Substitute this back into (2) to finish the proof of Claim 2. □

We now finish the proof of the theorem. By Claim 2,

$$
\begin{aligned}
\sum_{j=1}^{n} \langle v_i, v_j \rangle^2 &= \|v_i\|^4 + \sum_{v_j \in N(v_i)} \langle v_i, v_j \rangle^2 + O(k^{2/3} d^{-1/3}) \\
&= n O(k^{-2/3} d^{-2/3}) + O(k^{2/3} d^{-1/3}) \quad \text{by Claim 1.}
\end{aligned}
$$

Also by Claim 1, $\sum_{i=1}^{n} \|v_i\|^2 = \Omega(n)$. Therefore, by Lemma 1,

$$
d \ge \operatorname{rank}(A) \ge \frac{\left(\sum_{i=1}^{n} \|v_i\|^2 \right)^2}{\sum_{i,j=1}^{n} \langle v_i, v_j \rangle^2} = \frac{\Omega(n^2)}{n \big(n O(k^{-2/3} d^{-2/3}) + O(k^{2/3} d^{-1/3}) \big)},
$$

hence $n = O(n k^{-2/3} d^{1/3}) + O(k^{2/3} d^{2/3})$. Since $O(k^{-2/3} d^{1/3}) = o(1)$, it follows that $|N| = n = O(k^{2/3} d^{2/3})$. Recalling (1), we obtain that

$$
|V| = |N| + |V \setminus N| = O(k^{2/3} d^{2/3}) = O(d^{4/3}).
$$

□

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