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K-CENTERPOINTS CONJECTURES FOR POINTSETS IN $\mathbb{R}^d$

NABIL H. MUSTAFA*
Université Paris-Est,
LIGM, Equipe A3SI, ESIEE Paris.
mustafan@esiee.fr

SAURABH RAY †
Computer Science, NYU, Abu Dhabi.
saurabh.ray@nyu.edu

MUDASSIR SHABBIR ‡
Dept. of Computer Science,
Information Technology University, Lahore.
mudassir.shabbir@itu.edu.pk

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ABSTRACT

In this paper, we introduce the notion of *k-centerpoints* for any set P of n points in $\mathbb{R}^d$. These unify and generalize previous results such as the classical centerpoint theorem [1], and the recently-proven ray-shooting theorem [2]. We define two variants: *affine* k-centerpoints, and *topological* k-centerpoints. We prove their equivalence in $\mathbb{R}^2$, and conjecture that these are in fact equivalent in any dimension. We present the first non-trivial bounds for these problems in $\mathbb{R}^d$, as well as present several conjectures related to them.

Keywords: Centerpoint; data depth; Ray Shooting Depth

1. Introduction

In this paper, we propose a simple set of conjectures which unify and generalize several previous results. We solve these conjectures for $\mathbb{R}^2$, present first non-trivial bounds in higher dimensions, and argue for why they might be true in full generality.

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Finally, via more detailed arguments, derive better bounds in $\mathbb{R}^3$. We first explain the background needed to understand the conjectures and our results.

Tukey Depth. One of the classical results in Discrete Geometry is the *Centerpoint Theorem* [1], which states the following: given any set P of n points in $\mathbb{R}^d$, there exists a point $q \in \mathbb{R}^d$, not necessarily in P, such that any closed halfspace containing q contains at least $n/(d+1)$ points of P. Any such point q is called a *centerpoint* of P. Furthermore, this is optimal. The Tukey-depth of a point q is the minimum number of points contained in any half-space containing q. The point of highest Tukey-depth w.r.t. P is called the *Tukey* median of P, and its depth called the Tukey depth of P.

This theorem and its generalizations has found several applications in combinatorial geometry, statistics, geometric algorithms and related areas [3,4,5, 6,7,8].

One way to view the Centerpoint Theorem is as a generalization of the concept of the *median* of a set P of n numbers in $\mathbb{R}$: when $d = 1$, the Centerpoint Theorem gives a point $q \in \mathbb{R}$ which has at least $\lfloor n/2 \rfloor$ numbers of P larger and $\lfloor n/2 \rfloor$ numbers smaller than q, i.e., the median of P. Therefore the centerpoint of a set P can essentially be viewed as one measure to capture the statistical data depth of P (e.g., the *centroid* is another measure, which can be seen as the generalization of *mean* of a set of numbers).

Simplicial Depth. Another closely related measure of data depth is the *Simplicial depth* of a point, defined as follows. (For the moment, we restrict ourselves to the two-dimensional case - as we'll see later, things are much less well-understood even in $\mathbb{R}^3$). Given a set P of n points in $\mathbb{R}^2$, the *simplicial depth* of a point q is the number of triangles spanned by P that contain q. The simplicial depth of P is the highest such depth of any point q. The First-selection Lemma (see [9], pg. 207) in two-dimensions states that any set P in $\mathbb{R}^2$ has simplicial-depth at least $n^3/27$. This was first proved in 1984 by Boros-Furedi [10] (see Bukh [11] for the 'Book Proof') and was also shown to be optimal [12].

Not surprisingly, there is a close relation between Tukey and Simplicial depth. This can be proved easily using the technique in [10] (Proof is easy and we leave it as an exercise for the reader):

Claim 1.1. *Given a set P of n points in $\mathbb{R}^2$ with Tukey depth τn, $1/3 \le \tau \le 1/2$, the simplicial depth of P is at least $(-2\tau^3 + 2.5\tau^2 - \tau + 1/6) \cdot n^3$.*

In particular, the above claim together with the centerpoint theorem implies the First-selection Lemma in $\mathbb{R}^2$.

In fact, apart from the above relation between depths of *pointsets*, there is also a relation between depth of *single* points:

[12], [13]. Any point q of Tukey depth τn has Simplicial depth at least $\frac{3\tau^2 - 4\tau^3}{3!} \cdot n^3 - O(n^2)$ [13]. This bound is tight [12]: for every τ and n, there exists a set P of n

points, and a point q of depth τn where the simplicial depth of q is at most the above lower-bound (upto lower-order terms).

Ray-Shooting Depth. Recently, an elegant new result has been discovered that easily implies both the Centerpoint theorem and the Simplicial depth theorem in the plane (it surprises the authors that such fundamental results are still being discovered!). Given P, let E be the set of all $\binom{n}{2}$ edges spanned by points of P. Then *Ray-Shooting depth* of a point $q \in \mathbb{R}^2$ is the smallest r such that any half-infinite ray from q in any direction $u \in \mathbb{S}^1$ intersects at least r edges in E. The Ray-Shooting depth (henceforth called RS-depth) of P is the maximum RS-depth of any point in $\mathbb{R}^2$. Then the Ray-Shooting Theorem states the following:

[2]. Any set P of n points in $\mathbb{R}^2$ has RS-depth at least $n^2/9$. Furthermore, this bound is tight.

Now let q be a point with RS-depth rn^2. Consider any line l through q. Then l must intersect at least $2rn^2$ edges, and therefore both halfspaces defined by l must contain at least $(n - n\sqrt{1-8r})/2$ points. For simplicial depth, consider, for each point $p \in P$, the ray from q in the direction $\vec{pq}$. Then for every edge $\{p_i, p_j\}$ that interesects this ray, the triangle defined by $\{p, p_i, p_j\}$ must contain q. Summing up these triangles over all points, each triangle can be counted three times, and so q lies in at least $rn^3/3$ distinct triangles. Let us state this direct connection:

Fact 1. Given a set P of n points in $\mathbb{R}^2$, and a point q with RS-depth rn^2. Then q has

- Tukey depth at least $n \cdot (1 - \sqrt{1-8r})/2$
- Simplicial depth at least $rn^3/3$.

Now, by Ray-Shooting theorem, there exists a point with RS-depth $n^2/9$. Plugging $r = 1/9$ gives the Centerpoint theorem and the First-selection Lemma. So *any* point with RS-depth at least $n^2/9$ is both a centerpoint, as well as a high simplicial-depth point.

The status in $\mathbb{R}^3$ and higher dimensions. Things are much less well-understood already in $\mathbb{R}^3$. Of the depth measures, as stated earlier, the Centerpoint theorem again gives the Tukey-depth bound of $n/4$ in $\mathbb{R}^3$, and this is optimal.

However, the optimal bound for the Simplicial-depth is not known. It is known that for any set P of n points in $\mathbb{R}^d$, there exists a point lying in $c_d \cdot n^{d+1}$ simplices, where c_d is a constant that depends on the dimension d. The determination of the exact value of c_d is a long-standing open problem. Bárány [14] proved that $c_d \geq \frac{1}{d!(d+1)^{d+1}}$. Bárány's bound was improved to $\frac{d^2+1}{(d+1)!(d+1)^{d+1}}$ by Wagner [13], who in fact showed that any point of depth τn is contained in at least the following number of simplices:

$$\frac{(d+1)\tau^d - 2d\tau^{d+1}}{(d+1)!} \cdot n^{d+1} - O(n^d) \tag{1}$$

More recently, Bukh, Matoušek and Nivash [12] showed an elegant construction of a point set P so that no point in $\mathbb{R}^d$ is contained in, upto lower-order terms, more than $(n/(d+1))^{d+1}$ simplices defined by P. Furthermore, they conjecture that this is the right bound.

For $d = 3$ then, the conjectured bound is $c_3 = 0.0039$. For this case, the bound of Wagner was recently improved by Basit *et al.* [15], where they showed that $c_3 \geq 0.0023$. This was further improved by a complicated topological argument by Gromov [16], who showed that $c_d \geq 2d/((d+1)(d+1)!^2)$. For $d = 3$ this gives $c_3 \geq 0.0026$. This bound for $\mathbb{R}^3$ has since been improved even further by Krall *et al.* [17] to $c_3 \geq 0.0031$.

In higher dimensions, the notion of RS-depth corresponds to finding a point q such that any half-infinite ray from q intersects "lots" of $(d-1)$-simplices spanned by points of P. No combinatorial bounds on the RS-depth of such a point are known for $d \geq 3$. It was not studied in the paper that proposed it [2], and their topological technique used for the two-dimensional case fails for $d = 3$ and above. Using the bounds on Simplicial-depth, it is not too hard to derive a first such bound: any set P of n points in $\mathbb{R}^d$ has RS-depth at least $2d/((d+1)((d+1)!)^2) \cdot n^d$.

Organization. In Section 2 we present a broad and uniform way of seeing much of the earlier work on geometric data-depth as part of a bigger simpler geometric structure. This geometric structure is fully proven in two dimensions and asymptotically optimally proven for $\mathbb{R}^d$ in Section refsec:2d. Finally we prove a near-optimal exact result for three dimensions in Section 4.

2. A Uniform View of Data Depth

Given the lack of optimal bounds for Simplicial-depth and RS-depth in dimensions higher than two, the first question is what are the bounds to expect? What would be a good conjecture?

We think that one good way to answer such questions is by the following analogy. Consider the following pointset of size n: take any simplex in $\mathbb{R}^d$ (does not have to be regular) and place a tiny cloud of $n/(d+1)$ points around each of its $(d+1)$ vertices. Call such a pointset a *Simplex-like* point set. For the kind of questions considered in this paper, this seems to represent the 'extremal' case. In other words, if some 'affine property' is true for this pointset, then it is true for any pointset.

So let's consider all these depth measures for the Simplex-like pointset over a simplex S. Take the centroid c of this simplex. Then the Centerpoint theorem follows because any halfspace containing c must contain at least one vertex of S, and so contains $n/(d+1)$ points. Similarly, each point from one of the vertices makes a simplex containing c, and so c is contained in $(n/(d+1))^{d+1}$ simplices. Finally, any half-infinite ray from c must intersect at least one facet of S, and so intersect $(n/(d+1))^d$ $(d-1)$-simplices spanned by P and so have that much RS-depth.

Let us give some justification for reliance on this extremal case. First note that, so far, all theoretical and empirical evidence seems to show that as the Tukey depth of a pointset increases, so does its Simplicial-depth and also its RS-depth. So, for example, for a pointset with the Tukey-depth the maximum of $n/2$, we get the maximum possible values of both Simplicial-depth and RS-depth. Theoretical results for $\mathbb{R}^2$ relating Tukey-depth to Simplicial-depth have already been discussed in Claim 1.1.

Second, below we outline an argument showing that when the Tukey-depth of a pointset is the lowest possible, i.e., $n/(d+1)$, then we get exactly the bounds one expects from the Simplex-like pointset. This, together with the first point, leads us to suspect that the bounds derived from the Simplex-like pointset might indeed be always realizable for any pointset.

So consider the following theorem of Boros-Furedi [10]: Given a set P of n points in $\mathbb{R}^d$ with Tukey-depth $n/(d+1)$, there exists a point p with depth $n/(d+1)$, and a set $\mathcal{H}$ of $d+1$ halfspaces $\{h_1, \ldots, h_{d+1}\}$, such that i) $|h_i \cap P| = n/(d+1)$, ii) p lies on the boundary plane of each h_i, and iii) $h_1 \cup \ldots \cup h_{d+1}$ cover the entire $\mathbb{R}^d$. It is easy to see that in this configuration, with the given constraints, the $d+1$ regions $A_i = H_i \bigcap \left(\cap_{j \neq i} \overline{H}_j\right)$ each contain exactly $n/(d+1)$ points, and all the other $2^{d+1}-2$ regions are empty. And it is not too hard to prove that in such a configuration in $\mathbb{R}^d$, the point p has centerpoint-depth $n/(d+1)$, it has simplicial-depth $(n/(d+1))^{d+1}$, and has RS-depth $(n/(d+1))^d$. In some sense, with respect to the point p, the n points are essentially in a Simplex-like position, combinatorially, if Tukey-depth is $n/(d+1)$.

Furthermore, the intuition one gets from Simplex-like pointsets corresponds to every information we know about these problems. It gives exactly the results known for $\mathbb{R}$ (which is trivial) and for $\mathbb{R}^2$. And it matches the conjecture in [12] that there always exists a point of simplicial depth $(n/(d+1))^{d+1}$ (ignoring lower-order terms).

Line-depth in $\mathbb{R}^3$. Let us continue with the consideration of the set P in $\mathbb{R}^3$ where $n/4$ points are placed near each of the vertices of some tetrahedron S. And let c be the centroid of S.

Then by considering the 3-dimensional space defined by a halfspace with c on its (2-dimensional) boundary, we get the notion of Centerpoints. By considering the 1-dimensional space defined by a half-line with c on its (0-dimensional) boundary, we get the notion of RS-depth. But this begs the question: what about 2-dimensional space with c on its (1-dimensional) boundary? The natural answer is to consider the 2-dimensional space defined by a half-plane h with c on its (1-dimensional) boundary. And then count the number of *edges* spanned by P that intersect h. And what answer is to be expected? Going by the intutition of Simplex-like pointset, any half-plane through c will intersect at least one edge of S, and so intersect at least $(n/4)^2$ edges spanned by P. Formally, a point $q \in \mathbb{R}^3$ has *Line-depth* r if any halfplane through q intersects at least r edges spanned by P. The Line-depth of a pointset P is the highest Line-depth of any point. We conjecture:

Conjecture 2.1. *Any set P of n points in $\mathbb{R}^3$ has Line-depth at least $(n/4)^2$.*

Like RS-depth and Simplicial-depth in $\mathbb{R}^3$, it seems hard to prove this exact bound using current techniques. But we are able to show the following (the rather lengthy proof of this is given in Section 4):

Theorem 1 (Proof in Section 4). *Given any set P of n points in $\mathbb{R}^3$, there exists a point q such that any halfplane through c intersects at least $2n^2/49$ edges spanned by P.*

So with the notion of Line-depth, we have three measures in $\mathbb{R}^3$ for any point q: 2-dimensional space is the familiar Tukey-depth, 1-dimensional space gives Line-depth, and 0-dimensional space gives RS-depth. Intuitively, it is clear that as the dimension of the flat decreases, the degrees of freedom increase and the problem becomes more complicated. On one end, optimal results for the 2-dimensional case (Tukey-depth) are known. And on the other end, very partial results are known for the 0-dimensional case. It is our hope that the middle 1-dimensional case will be more within current reach than the 0-dimensional case. That is another motivation to study the Line-depth problem.

3. K-centerpoints In $\mathbb{R}^2$ and $\mathbb{R}^d$ for $d \geq 4$

The previous view can be extended to d-dimensions, giving the following "affine k-centerpoints" conjectures:

Conjecture 3.1 (Affine k-centerpoints Conjectures). *Given a set P of n points in $\mathbb{R}^d$, and an integer $0 \leq k \leq d-1$, there exists a point $q \in \mathbb{R}^d$ (*a k-centerpoint*) such that any $(d-k)$-half flat through q intersects at least $(n/(d+1))^{k+1}$ k-simplices spanned by P.*

The case $k = 0$ is the centerpoint theorem in any $\mathbb{R}^d$. The case $d = 2, k = 1$ is the RS-depth result of [2].

The case $d = 3, k = 1$ is the Line-depth theorem, for which we have presented Theorem 1.

For the general k-centerpoints problems, one can get the following bounds by extending the technique of Bárány [14] and using the result of Gromov [16]. Note that this implies that for every k, d, there exists a point q and a constant $c_{d,k}$ such that any $(d-k)$-half flat through q intersects at least $c_{d,k} \cdot n^{k+1}$ k-simplices spanned by P. Using some approximations for binomial coeffcient and factorials, it can be showed that first bound dominates the expression below for $k < 0.9d$ and for $k > 0.9d$ the second bound starts to dominate for $d > 800$. For smaller values of d, this threshold tends to grow with d from $0.8d$ to $0.9d$.

Theorem 2. *Given a set* P *of* n *points in* $\mathbb{R}^d$*, and an integer* $0 \le k \le d-1$*, there exists a point* $q \in \mathbb{R}^d$ *such that any* $(d-k)$*-half flat through* q *intersects at least*

$$\max\left\{ \binom{n/(d+1)}{k+1}, \quad \frac{2d}{(d+1)(d+1)!\binom{n}{d-k}} \cdot \binom{n}{d+1} \right\}$$

k-simplices spanned by P*.*

Proof. Given P, use Tverberg's Theorem [9] to partition P into $t = n/(d+1)$ sets $P_1, \ldots, P_t$ such that there exists a point q in the convex hull of the points in P_i for all i. Consider any $(d-k)$-dimensional half-flat $\mathcal{F}$ through q, where $\partial\mathcal{F}$, the boundary of $\mathcal{F}$, is a $(d-k-1)$-dimensional flat containing q. Project $\mathcal{F}$ onto a $(k+1)$-dimensional subspace $\mathcal{H}$ orthogonal to $\partial\mathcal{F}$ such that the projection of $\mathcal{F}$ is a ray r in $\mathcal{H}$, and $\partial\mathcal{F}$ and q are projected to the point q'. And let $P'_1, \ldots, P'_t$ be the projected sets whose convex-hulls now contains the point q'. Then note that the k-dimensional simplex spanned by $(k+1)$ points $Q' \subset P'$ intersects the ray r if and only if the k-dimensional simplex defined by the correponding set Q in $\mathbb{R}^d$ interescts the flat $\mathcal{F}$. Now apply the single-point version [a] of Colorful Caratheodory's Theorem [14] to every $(k+1)$-tuple of sets, say $P'_1, \ldots, P'_{k+1}$, together with the point s at infinity in the direction antipodal to the direction of r to get a 'colorful' simplex, defined by s and one point from each P'_i, and containing q'. Then the ray r must intersect the k-simplex defined by the $(k+1)$ points of P', and so the corresponding points of P in $\mathbb{R}^d$ span a $(k+1)$-simplex intersecting $\mathcal{F}$. In total, we get $\binom{n/(d+1)}{k+1}$ k-simplices intersecting $\mathcal{F}$.

Another way is to use the result of Gromov [16], that given any set P of n points in $\mathbb{R}^d$, there exists a point q lying in $2d/((d+1)(d+1)!) \cdot \binom{n}{d+1}$ d-simplices. Now take any $(d-k)$-half flat through q. It must intersect at least one k-simplex of each d-simplex containing it, and where each k-simplex is counted at most $\binom{n}{d-k}$ times. And we get

$$\frac{2d}{(d+1)(d+1)!\binom{n}{d-k}} \cdot \binom{n}{d+1}$$

distinct k-simplices intersecting any $(d-k)$-half flat through q. □

Coming back to the Simplex-like pointset, one can further observe another thing. Say S is the tetrahedron of the Simplex-like point set in $\mathbb{R}^3$. And we're studying the Line-depth of the centroid c. Then, as mentioned earlier, any half-plane through c intersects at least one edge of the tetrahedron, so intersects at least $(n/4)^2$ edges spanned by P. But, in fact, something stronger is true: take any line l through c and move l in any way to 'infinity' (i.e., outside the convex-hull of P). Then it

[a]Given any point $s \in \mathbb{R}^d$ and d sets $P_1, \ldots, P_d$ in $\mathbb{R}^d$ such that each convex hull of P_i contains the origin, there exists a d-simplex spanned by s and one point from each P_i which also contains the origin.

still has to intersect at least one edge of the tetrahedron. So the property for the Simplex-like point set is in fact topological in nature.

In fact, this property is already true for centerpoints: if a point q has Tukey-depth r, then any plane through q has to cross at least r points to reach a point at infinity, regardless of whether the movement is any arbitrary continuous movement or affine. See [18] for a discussion of this.

The "affine" k-centerpoints conjectures can therefore be strengthened to a more natural topological version:

Conjecture 3.2 (k-centerpoints Conjectures). *Given a set P of n points in $\mathbb{R}^d$, and an integer $0 \leq k \leq d-1$, there exists a point $q \in \mathbb{R}^d$ such that any $(d-k-1)$-flat through q must cross at least $(n/(d+1))^{k+1}$ k-simplices spanned by P to move to a point at infinity.*

We now give a proof of these topological k-centerpoints conjectures in $\mathbb{R}^2$.

Theorem 3. *For any set P of n points in $\mathbb{R}^2$, the k-centerpoints conjectures are true.*

Proof. As centerpoint-depth is already proven to be topological, we only have to resolve the RS-depth case [b].

We will actually prove the contrapositive: given a set P of n points in $\mathbb{R}^2$, let q be the point with RS-depth ρ. Take any curve γ from q that intersects at most ρ edges spanned by P. We want to show that there exists a half-infinite ray from q that also intersects at most ρ edges spanned by P. We prove this in two steps: first by replacing γ by a piecewise linear curve, and then replacing this piecewise linear curve with a half-infinite ray.

Given any curve γ, let $RS(\gamma)$ denote the number of edges spanned by P that γ intersects.

Lemma 2. *Given a curve γ that starts at q and ends at the point at infinity, there exists a piecewise linear curve γ' starting at q and ending at a point at infinity, such that $RS(\gamma) = RS(\gamma')$.* □

Proof. Given P, let $\mathcal{A}$ be the arrangement induced by $\Theta(n^2)$ lines supporting all the edges spanned by points of P. The curve γ enters and leaves a number of cells in the arrangement $\mathcal{A}$. Lets say it enters some cell C at point s_i and then leaves that cell at point e_i. Replace this portion of γ between s_i and e_i by the straight-line edge $s_i e_i$ (by convexity of C, this lies completely inside C). Note that γ cannot intersect any edge in the interior of any cell C (no edge can intersect a cell of this arrangement). Therefore, the RS-depth does not change by this replacement.

[b]It has been communicated to us by an insightful reviewer that this was implicit in Gromov's paper [16]. However the techniques there are highly non-trivial and based on algebraic topology arguments, while our short proof is elementary.

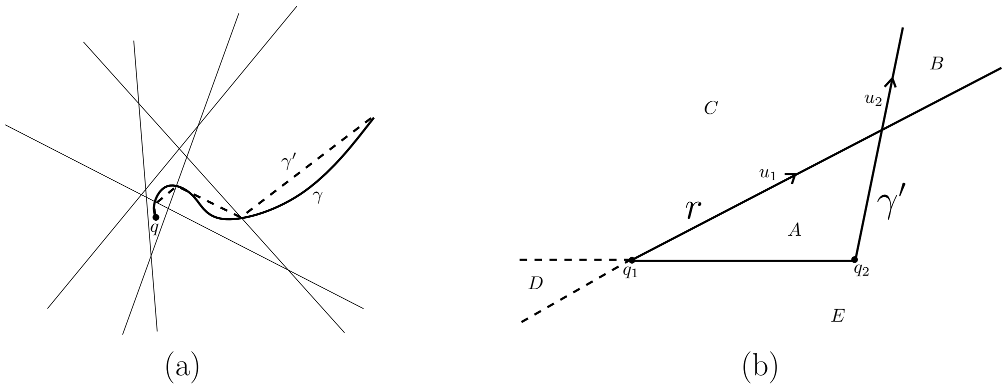

Fig. 1. (a) Converting γ to piecewise linear, (b) Illustration for Lemma 3.

Repeating this for each cell that γ enters and leaves, we get a sequence $\langle q = s_1, \ldots, s_m, u_1\rangle$, $s_i \in \mathbb{R}^2, u_1 \in \mathbb{S}^1$, which represents the piecewise linear curve γ' defined by the segments $s_1s_2, \ldots, s_{m-1}s_m$ together with the half-infinite ray $\vec{s}_m$ starting from s_m in the direction u_1. Finally, as discussed above, we have $RS(\gamma) = RS(\gamma')$. See Figure 1(a). □

Let γ' be the piecewise linear curve defined by the sequence $\langle s_1, \ldots, s_m, u_1\rangle$ as above. Consider the one-bend curve γ'' starting at s_{m-1} and defined by the segment $s_{m-1}s_m$ together with the half-infinite ray $\vec{s}_m$ in direction u_1. The Lemma below shows that there exists a direction $u_2 \in \mathbb{S}^1$ such that the half-infinite ray r starting at s_{m-1} in direction u_2 has $RS(r) \leq RS(\gamma'')$. In other words, γ'' can be 'straightened' to a ray r without increasing the number of edges intersected. This implies that

$$RS(\langle s_1, \ldots, s_{m-1}, u_2\rangle) \leq RS(\langle s_1, \ldots, s_m, u_1\rangle)$$

We now repeat this for the curve defined by $\langle s_1, \ldots, s_{m-1}, u_2\rangle$ to get another curve with one fewer bend. And so on till we get a ray r starting at q in direction $u \in \mathbb{S}^1$. By induction,

$$RS(\langle q = s_1, u\rangle) \leq RS(\langle s_1, s_2, u_{m-1}\rangle) \leq RS(\langle s_1, s_2, s_3, u_{m-2}\rangle) \leq \cdots \leq RS(\langle s_1, \ldots, s_m, u_1\rangle)$$

and the proof of Theorem 3 is completed. It remains to prove the following.

Lemma 3. *Given a piecewise linear curve γ' defined by $\langle q_1, q_2, u_2\rangle$, there exists a direction $u_1 \in \mathbb{S}^1$ such that $RS(\langle q_1, u_1\rangle) \leq RS(\langle q_1, q_2, u_2\rangle)$.*

Proof. See Figure 1(b). First we show the following:

Claim 3.1. *There exists a direction $u_1 \in \mathbb{S}^1$ such that the ray from q_1 in the direction u_1: i) intersects the ray from q_2 in direction u_2, ii) number of points in region A are equal to the number of points in region B (Figure 1(b)).* □

Proof. Start with the half-infinite ray r from q_1 through q_2. And continuously rotate r towards the direction u_2. Initially, one region is empty, while the other region contains the maximum points. When r points in the direction u_2, then the opposite is true. By intermediate-value theorem, there must be a ray in the middle that has equal number of points in both the region A and B (which could be 0). □

Let r be this ray from q_1 in direction u_1. And assume the regions A and B each contain t points. We now prove that this is the required ray, by showing that the number of edges intersected by r is at most those intersected by the curve γ'.

Edges that intersect both r and γ' or none, contribute equally, and so can be ignored. Consider the remaining edges spanned by P.

Observation 3.1. Any edge $e = \{p_i, p_j\}$ that intersects r , but not γ' , must have exactly one endpoint in region A or B.

Proof. We prove the contrapositive. If e has one endpoint in A and the other in B, then it must cross both r and γ'. On the other hand, if none are in A or B, then either e intersects both, or none. □

Therefore consider each point p_i not in A or in B (see Figure 1(b)):

- $p_i \in C$. Then p_i has exactly t edges intersecting γ' (to points in B), and exactly t edges intersecting r (to points in A).
- $p_i \in D$. Then p_i has at least t edges intersecting γ' (to points in B and possibly in A) and at most t edges intersecting r (to points in A).
- $p_i \in E$. Then p_i has at least t edges intersecting γ' (to points in A and possibly in B) and exactly t edges intersecting r (to points in B).

Summing up over all p_i proves the Lemma.

4. *K*-centerpoints In $\mathbb{R}^3$

For $\mathbb{R}^3$, besides the already studied Tukey-depth, Simplicial depth, and Ray-shooting depth, we have proposed the problem of Line-depth. In this section we give improved bounds for it. For any set P with n points and Tukey-depth τn, our bound is achieved via a two-step strategy. First, we show that there exists an increasing function of τ that lower-bounds the Line-depth of P. Then, via an alternate technique, we show the existence of a decreasing function of τ that also lower-bounds the Line-depth of P. Combining the two yields our theorem.

Lemma 4. *Given a set P of n points in $\mathbb{R}^3$, let p be a point with Tukey depth at least τn. Then the Line-depth of p is at least $(\tau n)^2/2 - o(n)$.*

Proof. Given an arbitrary half-plane H through such a point p, we define a procedure to find edges that intersect H. Starting from H, rotate a half-plane in one direction with the axis of rotation as the bounding line of H. Sort points in P by the order in which they intersect this rotating half-plane, i.e., p_1 is the first point to be hit. Call the half-plane through p_i as H_i, and the plane containing H_i as G_i. Let H_i^+ be the halfspace defined by G_i such that $H \subseteq H_i^+$. H partitions H_i^+ into H_i^{++} and H_i^{+-}; denote the wedge containing points p_1 through p_i as H_i^{++}. By definition of depth of p, $|H_i^{++} \cap P| + |H_i^{+-} \cap P| \geq \tau n$. Since $|H_i^{++} \cap P| = i$, we have $|H_i^{+-} \cap P| \geq \tau n - i$. Observe that for any $i \leq \tau n$, the line segment defined by p_i and $p_j \in H_i^{+-} \cap P$ must intersect H. Number of such line segments can be bounded as

$$T \geq \sum_{k=1}^{\tau n} k = \frac{(\tau n)^2}{2} - o(n) \tag{2}$$

□

Note that since $\tau \geq 1/4$ by the Centerpoint theorem, Lemma 4 proves the existence of a point with Line-depth at least $n^2/32$. By the second method, we prove the following lower-bound:

Lemma 5. *Given a set P of n points with Tukey-depth τn, there exists a point q with Line-depth at least $(\tau - 3\tau^2) \cdot n^2$.*

Note that this is a decreasing function of τ for $\tau \in [0.25, 0.5]$. For the proof, we extend the approach in [15] to work for the Line-depth case. First a key lemma.

Lemma 6 (Boros-Füredi [10]). *Given a set P of n points in $\mathbb{R}^d$, where depth$(P) = \tau n$, there exists a point p with depth τn, and a set $\mathcal{H}$ of $d+1$ halfspaces $\{h_1, \ldots, h_{d+1}\}$, such that i) $|h_i \cap P| = \tau n$, ii) p lies on the boundary plane of each h_i, and iii) $h_1 \cup \ldots \cup h_{d+1}$ cover the entire $\mathbb{R}^d$.*

Given a set P of n points in $\mathbb{R}^3$, with *depth*$(P) = \tau n$, use Lemma 6 to get the point p and a set of four halfspaces $\{h_1, h_2, h_3, h_4\}$ satisfying the stated conditions. The point p, and halfspaces $\{h_1, \ldots, h_4\}$ will refer to these for the rest of the proof. We will now show that p gives the required Line-depth lower-bound of $(\tau - 3\tau^2) \cdot n^2$. These four halfspaces partition $\mathbb{R}^3$ into the following convex unbounded regions:

$$A_i = (\bigcap_{l \neq i} \overline{h}_l) \cap h_i, \qquad B_{i,j} = (\bigcap_{l \neq i,j} \overline{h}_l) \cap h_i \cap h_j, \qquad C_i = (\bigcap_{l \neq i} h_l) \cap \overline{h}_i \tag{3}$$

See Figure 2. We note that regions A_i and C_i are *antipodal* around the point p (in the sense that a line through p and intersecting A_i will intersect C_i and not intersect any other region). Similarly region $B_{i,j}$ is antipodal to region $B_{k,l}$ for distinct $1 \leq i, j, k, l \leq 4$. For brevity we also define

$$A := \bigcup_{i \in [4]} A_i, \qquad B := \bigcup_{i,j \in [4], i \neq j} B_{i,j}.$$

Set $\alpha_i = |P \cap A_i|/n$, $\beta_{i,j} = |P \cap B_{i,j}|/n$, and $\gamma_i = |P \cap C_i|/n$. Note that we have these two constraints on the non-negative variables $\alpha_i, \beta_{i,j}$ and γ_i:

$$\tau = \frac{|h_i \cap P|}{n} = \alpha_i + \sum_{j \neq i} \beta_{i,j} + \sum_{j \neq i} \gamma_j \qquad \text{for each } i = 1 \ldots 4. \tag{4}$$

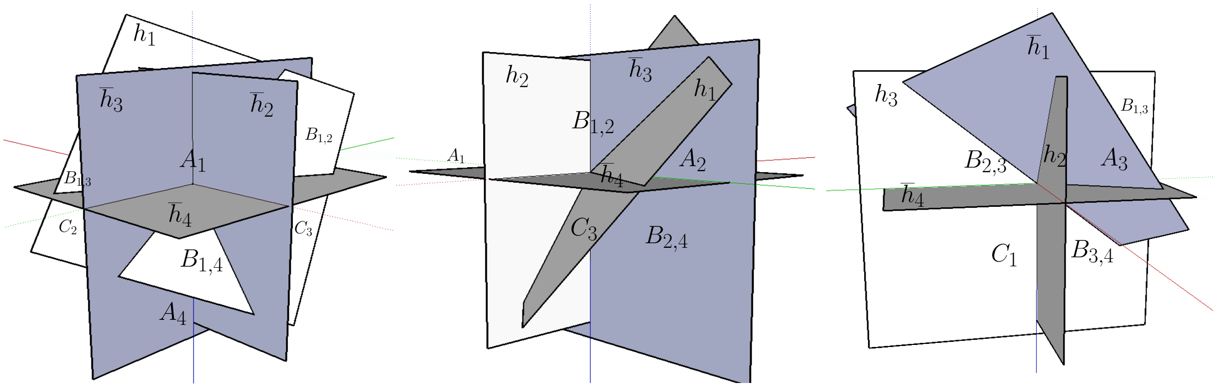

Fig. 2. Partitioning of $\mathbb{R}^3$ into 14 regions by the four halfspaces as visible from three different angles. The labels on the planes represent the halfspace that is facing the reader.

$$\sum_i \alpha_i + \sum_{i<j} \beta_{i,j} + \sum_i \gamma_i = 1, \qquad \text{as } \{h_1, h_2, h_3, h_4\} \text{ cover } \mathbb{R}^3 \setminus \{p\}. \tag{5}$$

Summing up (4) for all four halfspaces, and subtracting (5) from it, we get

$$\sum_{i<j} \beta_{i,j} + 2 \cdot \sum_i \gamma_i = 4 \cdot \tau - 1. \tag{6}$$

Therefore, $\sum_{i<j} \beta_{i,j} + \sum_i \gamma_i \leq 4\tau - 1$. This fact, together with equation (4), implies that $1 - 3\tau \leq \alpha_i \leq \tau$ for $i = 1 \ldots 4$. Furthemore, we need the following lemma. We will use $\overline{h}_i$ for complement of a halfspace h_i, i.e., $\overline{h}_i = \mathbb{R}^3 \setminus h_i$.

Lemma 7. *For any $1 \leq i < j \leq 4$, we have $\alpha_i \alpha_j + \beta_{i,j} \alpha_i \geq \tau - 3\tau^2$. Similarly, $\alpha_i \alpha_j + \beta_{i,j} \alpha_j \geq \tau - 3\tau^2$.*

Proof. Assume the other two planes are h_k and h_l (other than h_i and h_j). Since $|P \cap \overline{h}_k| = 1 - \tau$, and $|P \cap h_l| = \tau$, we get $P \cap (\overline{h}_k \setminus h_l) = \alpha_i + \beta_{i,j} + \alpha_j \geq 1 - 2\tau$. Then we have

$$\alpha_i \alpha_j + \beta_{i,j} \alpha_i \geq \alpha_i \alpha_j + (1 - 2\tau - \alpha_i - \alpha_j)\alpha_i = \alpha_i (1 - 2\tau - \alpha_i)$$

This last term is minimized at the extreme values of α_i, which are either τ or $1 - 3\tau$, both yielding a lower-bound of $\tau(1 - 3\tau)$. □

We repeatedly use the following fact in different cases below to count the number of line segments intersecting an arbitrary half plane H.

Fact 8. Given a set of halfspaces $\mathcal{H}$ in $\mathbb{R}^3$, let X be the convex-region of their common intersection. And let H be any set s.t. $X \setminus H$ has more than one path-connected component. Then if p and q are two points in two different components, the edge $\overline{pq}$ must intersect H.

Let A_i^+, A_i^- be a partition of A_i and $B_{i,j}^+$, $B_{i,j}^-$ a partition of $B_{i,j}$ (to be specified in individual cases later). Then define: $\alpha_i^+ = \frac{|A_i^+ \cap P|}{n}, \alpha_i^- = \frac{|A_i^- \cap P|}{n}, \beta_{i,j}^+ = \frac{|B_{i,j}^+ \cap P|}{n}, \beta_{i,j}^- = \frac{|B_{i,j}^- \cap P|}{n}$.

Claim 4.1. $\alpha_i \cdot \alpha_j + \beta_{i,j}^+ \cdot \alpha_j + \beta_{i,j}^- \cdot \alpha_i \geq \tau - 3\tau^2$.

Proof.

$$\begin{aligned}
\alpha_i \cdot \alpha_j + \beta_{i,j}^+ \cdot \alpha_j + \beta_{i,j}^- \cdot \alpha_i &\geq \alpha_i\alpha_j + \min(\alpha_i, \alpha_j)(\beta_{i,j}^+ + \beta_{i,j}^-) \\
&= \alpha_i\alpha_j + \beta_{i,j}\min(\alpha_i, \alpha_j) \\
&= \min(\alpha_i\alpha_j + \beta_{i,j}\alpha_i, \alpha_i\alpha_j + \beta_{i,j}\alpha_j) \geq \tau - 3\tau^2
\end{aligned}$$

by Lemma 7. □

Claim 4.2. $\alpha_i^+ \cdot \alpha_j + \alpha_i^- \cdot \alpha_k + \alpha_i^+ \cdot \beta_{i,j}^- + \beta_{i,j}^+ \cdot \alpha_j + \alpha_i^- \cdot \beta_{i,k} \geq \tau - 3\tau^2$.

Proof.

$$\begin{aligned}
&= \alpha_i^+(\alpha_j + \beta_{i,j}^-) + \alpha_i^-(\alpha_k + \beta_{i,k}) + \alpha_j\beta_{i,j}^+ \\
&\geq \min(\alpha_j + \beta_{i,j}^-, \alpha_k + \beta_{i,k})(\alpha_i^+ + \alpha_i^-) + \alpha_j\beta_{i,j}^+ \\
&= \alpha_i\min(\alpha_j + \beta_{i,j}^-, \alpha_k + \beta_{i,k}) + \alpha_j\beta_{i,j}^+ \\
&= \min(\alpha_i\alpha_j + \alpha_i\beta_{i,j}^- + \alpha_j\beta_{i,j}^+, \alpha_i\alpha_k + \beta_{i,k}\alpha_i + \alpha_j\beta_{i,j}^+) \\
&\geq \min(\alpha_i\alpha_j + \min(\alpha_i, \alpha_j)(\beta_{i,j}^+ + \beta_{i,j}^-), \alpha_i\alpha_k + \beta_{i,k}\alpha_i) \\
&= \min(\min(\alpha_i\alpha_j + \alpha_i\beta_{i,j}, \alpha_i\alpha_j + \alpha_j\beta_{i,j}), \alpha_i\alpha_k + \beta_{i,k}\alpha_i) \\
&= \min(\alpha_i\alpha_j + \alpha_i\beta_{i,j}, \alpha_i\alpha_j + \alpha_j\beta_{i,j}, \alpha_i\alpha_k + \beta_{i,k}\alpha_i) \\
&\geq \tau - 3\tau^2
\end{aligned}$$

where the last inequality follows from Lemma 7. □

Claim 4.3. $\alpha_i^- \cdot \alpha_l^+ + \alpha_i^+ \cdot \alpha_k + \alpha_j \cdot \alpha_l^- + \beta_{i,l}^+ \cdot \alpha_i^- + \beta_{i,l}^- \cdot \alpha_l^+ + \alpha_i^+ \cdot \beta_{i,k} + \alpha_l^- \cdot \beta_{j,l} \geq \tau - 3\tau^2$.

Proof.

$$\begin{aligned}
&= \alpha_i^+(\alpha_k + \beta_{i,k}) + \alpha_i^-(\alpha_l^+ + \beta_{i,l}^+) + \alpha_l^-(\alpha_j + \beta_{j,l}) + \alpha_l^+ \cdot \beta_{i,l}^- \\
&\geq \alpha_i \cdot \min(\alpha_k + \beta_{i,k}, \alpha_l^+ + \beta_{i,l}^+) + \alpha_l^-(\alpha_j + \beta_{j,l}) + \alpha_l^+ \cdot \beta_{i,l}^- \\
&\geq \min(\alpha_i \cdot \alpha_k + \alpha_i \cdot \beta_{i,k}, \alpha_i \cdot \alpha_l^+ + \alpha_i \cdot \beta_{i,l}^+ + \alpha_l^-(\alpha_j + \beta_{j,l}) + \alpha_l^+ \cdot \beta_{i,l}^-) \\
&\geq \min(\alpha_i \cdot \alpha_k + \alpha_i \cdot \beta_{i,k}, \alpha_l^+(\alpha_i + \beta_{i,l}^-) + \alpha_l^-(\alpha_j + \beta_{j,l}) + \alpha_i \cdot \beta_{i,l}^+) \\
&\geq \min(\alpha_i \cdot \alpha_k + \alpha_i \cdot \beta_{i,k}, \alpha_l\min(\alpha_i + \beta_{i,l}^-, \alpha_j + \beta_{j,l}) + \alpha_i \cdot \beta_{i,l}^+) \\
&\geq \min(\alpha_i \cdot \alpha_k + \alpha_i \cdot \beta_{i,k}, \min(\alpha_l \cdot \alpha_j + \alpha_l \cdot \beta_{j,l}, \alpha_l \cdot \alpha_i + \alpha_l \cdot \beta_{i,l}^- + \alpha_i \cdot \beta_{i,l}^+)) \\
&\geq \min(\alpha_i \cdot \alpha_k + \alpha_i \cdot \beta_{i,k}, \alpha_l \cdot \alpha_j + \alpha_l \cdot \beta_{j,l}, \alpha_l \cdot \alpha_i + \beta_{i,l}\min(\alpha_l, \alpha_i)) \\
&\geq \min(\alpha_i \cdot \alpha_k + \alpha_i \cdot \beta_{i,k}, \alpha_l \cdot \alpha_j + \alpha_l \cdot \beta_{j,l}, \alpha_l \cdot \alpha_i + \beta_{i,l} \cdot \alpha_l, \alpha_l \cdot \alpha_i + \beta_{i,l} \cdot \alpha_i) \\
&\geq \tau - 3\tau^2.
\end{aligned}$$

where last line follows from Lemma 7. □

Let p be as defined in Lemma 6; we will show that any half-plane through p intersects at least $(\tau - 3\tau^2) \cdot n^2$ edges spanned by P. We identify a line ℓ as the supporting line of the halfplane H if it lies on the boundary of H. The supporting line ℓ of H passes through a pair of antipodal regions among the fourteen regions define in Equation 3 as well as the point p.

There are two obvious possibilities for a halfplane H: either it intersects A (recall that A is the union of $A_i, i \in [4]$) or it does not intersect A (**Case 2** below). For the first possibility we can have three possible subcases: supporting line ℓ of H passes through for some A_i but H doesn't intersect $A \setminus A_i$ (**Case 1**) or H intersects A but the supporting line ℓ doesn't (**Case 4**) or ℓ passes through some A_i and H intersects some $A_j, j \neq i$ (**Case 3**).

The four halfspaces create the structure of a cuboctahedron. The bounding planes of the four halfspaces partition a sphere S centered on p into 8 triangular and 6 quadrilateral faces creating a cuboctahedron structure. A triangular face represents a spherical cross-section of either an A_i region (lying in exactly one halfspace h_i) or a C_i region (lying in all but one halfspace h_i). Similarly a quadrilateral face represents a spherical cross-section of a B_{ij} region (lying in exactly two halfspaces h_i and h_j). Each A_i is adjacent to all $B_{i,j}$ where $j \neq i$, and each C_i is adjacent to all $B_{j,k}$ such that $j, k \neq i$.

Now, let π be a fifth plane through p that intersects the other four planes generically. There is combinatorially only one way of doing this: The plane π intersects the following 8 faces of the cuboctahedron in circular order: $(A_i, B_{i,j}, C_k, B_{j,l}, C_i, B_{k,l}, A_k, B_{k,i})$.

We want a halfplane, not a full plane, so we choose a halfplane H of π bounded by a line ℓ that passes through p. Since ℓ intersects two opposite faces, the halfplane H can intersect any 5 adjacent faces out of the above circular list. Therefore, we get the following possibilities(ignoring the cases that are symmetric to the ones noted below): (a). $A_i, B_{i,j}, C_k, B_{j,l}, C_i$ (**Case 1**), (b). $B_{i,j}, C_k, B_{j,l}, C_i, B_{k,l}$ (**Case 2**), (c). $B_{j,l}, C_i, B_{k,l}, A_k, B_{k,i}$ (**Case 3**), (d). $C_i, B_{k,l}, A_k, B_{k,i}, A_i$ (**Case 4**), and (e). $B_{k,l}, A_k, B_{k,i}, A_i, B_{i,j}$ (**Case 4**).

It suffices to prove the bound on the number of edges intersecting a half-plane H, say denoted by $\eta(H)$, separately for the following four cases. We will use terms *above (or below) a half-plane* H to distinguish between points that lie on opposite sides of the plane passing through H.

Case 1. (**Supporting line ℓ passes through A_i and H does not intersect $A \setminus A_i$ for $i \in [4]$**).
Note that in this case the supporting line ℓ also passes through C_i. By definition $C_i = (\bigcap_{j \neq i} h_j) \cap \overline{h}_i$. Since there is no point common in all four (open) halfspaces (or

their complements) we also have that $C_i = (\bigcap_{j \neq i} h_j)$. There are three neighbors[c] of C_i: $B_{j,k}, B_{k,l}, B_{j,l}$ where $j, k, l \in [4] \setminus \{i\}$ and any halfplane H with supporting line passing through C_i will intersect exactly one of them (see Figure 3). Without loss of generality let us assume that H intersects some $B_{j,k}$ where $j, k \neq i$, and let $\mathcal{H} = \{\overline{h}_i, \overline{h}_l\}$. By definition, all the points in $A_j \cup B_{j,k} \cup A_k$ lie in the intersection of halfspaces in $\mathcal{H}$. As H does not intersect A_j and A_k (by assumption), H partitions the region defined by the intersection of halfspaces in $\mathcal{H}$ into two pieces, with $A_j \cup B^+_{j,k}$ lying above H, while $A_k \cup B^-_{j,k}$ lying below H. Using Fact 8 with $\mathcal{H}$ and H, we get that the following number of edges must intersect H: $\eta(H) \geq \alpha_j \cdot \alpha_k + \beta^+_{j,k} \cdot \alpha_k + \beta^-_{j,k} \cdot \alpha_j$. From Claim 4.1, it follows that $\eta(H) \geq \tau - 3\tau^2$.

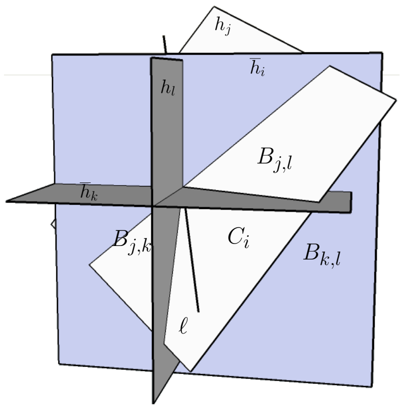

Fig. 3. Region C_i has three neighboring regions: $B_{j,k}, B_{j,l}$ and $B_{k,l}$ and for a halfplane H with supporting line ℓ passing through A_i and C_i, H intersects exactly one of them.

Case 2. (H **does not intersect any** A_i **for** $i \in [4]$).
If H does not intersect any A_i then it must be that the supporting line ℓ passes through $B_{i,j}$ and $B_{k,l}$ and H intersects B_{xy} with $x \in \{i, j\}$ and $y \in \{k, l\}$. With loss of generality assume that $x = j$ and $y = k$. Let $\mathcal{H} = \{\overline{h}_i, \overline{h}_l\}$. Then, by definition, all the points in $A_j \cup B_{j,k} \cup A_k$ lie in the intersection of halfspaces in $\mathcal{H}$. And in this case of H, H partitions the region defined by the intersection of halfspaces in $\mathcal{H}$ into two pieces, with $A_j \cup B^+_{j,k}$ lying above H, while $A_k \cup B^-_{j,k}$ lying below H. Using Fact 8 with $\mathcal{H}$ and H, we get that the following number of edges must intersect H: $\eta(H) \geq \alpha_j \cdot \alpha_k + \beta^+_{j,k} \cdot \alpha_k + \beta^-_{j,k} \cdot \alpha_j$. From Claim 4.1, it follows that $\eta(H) \geq \tau - 3\tau^2$.

Case 3. (**Supporting line** ℓ **passes through** A_i **and** H **also intersects some** A_j **for some** $i, j \in [4], j \neq i$).

[c]We call a pair of regions as neighbors (to each other) if exactly one halfspace h_i separates points in one region from the points in the other region.

In this case H may also intersect $B_{j,k}$ or $B_{j,l}$ but not both. Without loss of generality H intersects $B_{j,k}$ (if H does not intersect any of them then we can take either one part of the assumed partition to contain no points of P). Let $\mathcal{H} = \{\overline{h}_i, \overline{h}_l\}$. Then, by definition, all the points in $A_j \cup B_{j,k} \cup A_k$ lie in the intersection of halfspaces in $\mathcal{H}$. And in this case of H, H partitions the region defined by the intersection of halfspaces in $\mathcal{H}$ into two pieces, with $A_j^+ \cup B_{j,k}^+$ lying above H, while $A_j^- \cup A_k \cup B_{j,k}^-$ lying below H. Using Fact 8 with $\mathcal{H}$ and H, we get that edges with endpoints in the following pairs of regions are intersecting H: $(A_j^+, A_k), (A_j^+, B_{j,k}^-), (A_k, B_{j,k}^+)$. Similarly H also partitions $\overline{h}_i \cap \overline{h}_k$; setting $\mathcal{H} = \{\overline{h}_i, \overline{h}_k\}$, the edges with endpoints in following pairs must intersect H: (A_l, A_j^-), $(A_j^-, B_{j,l})$. Therefore, the following number of edges must intersect H: $\eta(H) \geq \alpha_j^+ \cdot \alpha_k + \alpha_j^- \cdot \alpha_l + \alpha_j^+ \cdot \beta_{j,k}^- + \beta_{j,k}^+ \cdot \alpha_k + \alpha_j^- \cdot \beta_{j,l}$.

From Claim 4.2, it follows that $\eta(H) \geq \tau - 3\tau^2$.

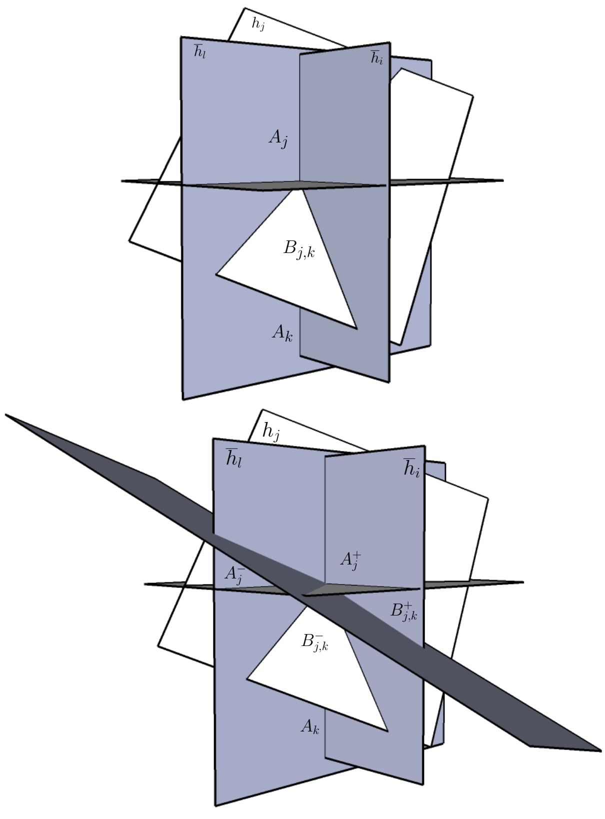

Fig. 4. Halfplane H partitions $\overline{h}_i \cap \overline{h}_l = A_j \cup B_{j,k} \cup A_k$ into two parts: $A_j^+ \cup B_{j,k}^+$ lying above H and $A_j^- \cup A_k \cup B_{j,k}^-$ lying below H.

Case 4. (H **intersects** A **but the supporting line** ℓ **doesn't**).
Since supporting line ℓ does not intersect A, it must intersect B. Without loss of any generality, ℓ passes through $B_{i,j}$ and $B_{k,l}$ and H intersects A_j. In this case H may also intersect $B_{j,k}$ or $B_{j,l}$ but not both. Again without loss of generality H intersects $B_{j,k}$ (if H does not intersect any of them then we can take either one part of the assumed partition to contain no points of P). There are two possible subcases (4.1) H intersects only one of A_j and A_k and (4.2) H intersects both of A_j and A_k.

(4.1). Without loss of generality H only intersects A_j. Let $\mathcal{H} = \{\overline{h}_i, \overline{h}_l\}$. Then, by definition, all the points in $A_j \cup B_{j,k} \cup A_k$ lie in the intersection of halfspaces in $\mathcal{H}$. And in this case of H, H partitions the region defined by the intersection of halfspaces in $\mathcal{H}$ into two pieces, with $A_j^+ \cup B_{j,k}^+$ lying above H, while $A_j^- \cup A_k \cup B_{j,k}^-$ lying below H. See Figure 4. Using Fact 8 with $\mathcal{H}$ and H, we get that edges with endpoints in the following pairs of regions are intersecting H: $(A_j^+, A_k), (A_j^+, B_{j,k}^-), (A_k, B_{j,k}^+)$. Similarly H also partitions $\overline{h}_i \cap \overline{h}_k$; setting $\mathcal{H} = \{\overline{h}_i, \overline{h}_k\}$, the edges with endpoints in following pairs must intersect H: (A_l, A_j^-), $(A_j^-, B_{j,l})$. Therefore, the following number of edges must intersect H: $\eta(H) \geq \alpha_j^+ \cdot \alpha_k + \alpha_j^- \cdot \alpha_l + \alpha_j^+ \cdot \beta_{j,k}^- + \beta_{j,k}^+ \cdot \alpha_k + \alpha_j^- \cdot \beta_{j,l}$.

From Claim 4.2, it follows that $\eta(H) \geq \tau - 3\tau^2$.

(4.2). Let $\mathcal{H} = \{\overline{h}_i, \overline{h}_l\}$. Then, by definition, all the points in $A_j \cup B_{j,k} \cup A_k$ lie in the intersection of halfspaces in $\mathcal{H}$. And in this case of H, H partitions the region defined by the intersection of halfspaces in $\mathcal{H}$ into two pieces, with $A_j^+ \cup +A_k^+ \cup B_{j,k}^+$ lying above H, and $A_j^- \cup +A_k^- \cup B_{j,k}^-$ lying below H. Fact 8 implies that line segments between following pairs of regions are intersecting H: $(A_j^+, A_k^-), (A_j^-, A_k^+), (A_j^-, B_{j,k}^+)$, $(A_k^-, B_{j,k}^+), (A_j^+, B_{j,k}^-), (A_k^+, B_{j,k}^-)$. Also, setting $\mathcal{H} = \overline{h}_i \cap \overline{h}_k$, line segments between following pairs must intersect H: (A_j^+, A_l), $(A_j^+, B_{j,l})$ and setting $\mathcal{H} = \overline{h}_j \cap \overline{h}_l$, line segments between following pairs must intersect H: (A_k^-, A_i), $(A_k^-, B_{i,j})$. Therefore, following number of edges must intersect H: $\eta(H) \geq \alpha_j^- \cdot \alpha_k^+ + \alpha_j^+ \cdot \alpha_l + \alpha_i \cdot \alpha_k^- + \beta_{j,k}^+ \cdot \alpha_j^- + \beta_{j,k}^- \cdot \alpha_k^+ + \alpha_j^+ \cdot \beta_{j,l} + \alpha_k^- \cdot \beta_{i,k}$.

From Claim 4.3, it follows that $\eta(H) \geq \tau - 3\tau^2$.

This completes the proof of Lemma 5.

Finally, we complete the proof of Theorem 1: take any set P of n points with Tukey-depth τn. If $\tau \geq 0.285$, Lemma 4 gives a point with Line-depth at least $(0.285)^2 n^2/2 \geq n^2/24.5$. On the other hand, if $\tau < 0.285$, Lemma 5 gives a point with Line-depth at least $(0.285 - 3(0.285)^2)n^2 \geq n^2/24.5$.

5. Discussion

Relations between data-depth measures in $\mathbb{R}^3$. For $d = 2$, we have two fundamental measures: Tukey-depth, and RS-depth. Recall from Fact 1 that any point of RS-depth $n^2/9$ has Tukey-depth at least $n/3$. On the other hand, a centerpoint is not always a point of "high" ray-shooting depth. Consider the example in the figure 5. The centerpoint c in the figure has ray-shooting depth at most $n^2/18 + 6n$. It can be shown that this is the worst possible example - that a centerpoint always

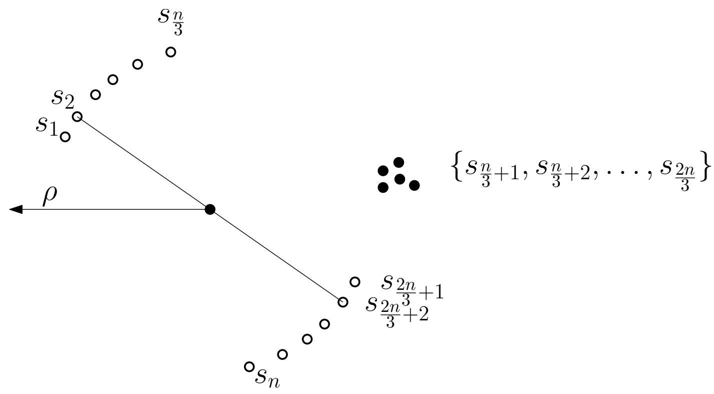

Fig. 5. Two thirds of the points are arranged on a unit circle around a point c so that the line through the pair $(s_i, s_{\frac{2n}{3}+i})$ also passes through c for all $1 \leq i \leq n/3$. Clearly c is a centerpoint but c doesn't have "high" ray-shooting depth as the ray ρ starting at c intersects at most $n^2/18 + 6n$ edges.

has ray-shooting depth of $n^2/18 - O(n)$.

Fact 9. Given a set of points S in the plane any centerpoint c has ray-shooting depth more than $n^2/18 - O(n)$.

Proof. It is enough to show that there are at the least acclaimed number of edges meeting an arbitrary ray ρ from the centerpoint c. Since a point in S contributes at most $n-1$ edges, we may assume that ρ meets a point s_1. Let the points in $S = \{s_1, s_2, \ldots, s_n\}$ be ordered radially around c in the counter-clockwise order. Translate and rotate the point set so that c is the origin and s_1 lies on the positive horizontal axis. Let ℓ_i be the line through c and the point s_i - so ℓ_1 is the horizontal axis with at least $n/3$ points above and at least $n/3$ points below. Note that for a point s_i with $i \leq n/3$ and a point s_j with $j \geq n - (n/3 - i)$, the edge $\overline{s_i s_j}$ meets the ray ρ. This is because there are at least $n/3$ points to the right of the line ℓ_i. And at least $n - (n/3 - i)$ of them lie below ρ. There are at least $(n/3 - i)$ edges for each s_i with $i \leq n/3$ that meet ρ. Therefore the number of edges meeting ρ is more than $\sum_{i=1}^{n/3}(n/3 - i)$. The claim follows. $\square$

So there is a hierarchy for $d = 2$. Is such a similar property true for $\mathbb{R}^3$?

Speculation 10. Let q be a point with RS-depth $(n/4)^3$. Then q has Line-depth at least $(n/4)^2$. Similarly, let q be a point with Line-depth $(n/4)^2$. Then q has Tukey-depth at least $(n/4)$.

Unfortunately, the hierarchical structure that is present in $\mathbb{R}^2$ is not true in $\mathbb{R}^3$.

Claim 5.1. *There exists a set of points P and a point q such that q has Line-depth* *at least* $(1/16 + 1/128)n^2 - cn$ *and Tukey-depth less than* $(n/4)$ *where* c *is a fixed constant independent of* n.

Proof. To prove this statement we will construct one such set of n points and a point q with claimed bounds on its line-depth and Tukey depth. Our pointset P has two parts A and B with $|A| = 3n/4$ and $|B| = n/4$. We place the points in A around the origin at regular distance in the unit radius circle in the XY-plane with $z = 0$. Similarly the points in B lie in a circle of small enough radius $\epsilon_r > 0$ around the point $(0, 0, 1)$ at regular distance in the plane $z = 1$[d]. We fix q to be the midpoint on the line segment between the centers of these two circles i.e. $q = (0, 0, 1/2)$. It is easy to see that the Tukey depth of q is at most $n/4$.

We want to show that for any halfplane H through q there are at least $(1/16 + 1/128)n^2$ edges incident on the pairs of points in P that intersect H. The halfplane H is defined by a supporting line ℓ and a direction δ. By symmetry of points in A and B we can assume, without loss of any generality, that the line ℓ lies on XZ-plane and that the direction δ lies in the positive Y halfspace. For a point u in the three dimensional space we use $x(u), y(u), z(u)$ to represent x, y, and z coordinates of u respectively. We write $ch(A)$ and $ch(B)$ for respective convex hulls of the points in A and B. For a set Q of points $\pi(Q)$ denotes the plane through Q whenever Q defines a unique such plane. Unless a halfplane misses both the convex hulls (which is described in Case 1), its supporting line ℓ may or may not intersect $ch(B)$. The scenario when it does intersect $ch(B)$ is described in Case 2 below. We divide the case when it does not intersect $ch(B)$ into further subcases based on whether ℓ intersect $ch(A)$ or not.

Case 1: H intersects neither ch(A) nor ch(B).

It is easy to see that for all $a \in A$ in positive Y halfspace (i.e. $y(a) \geq 0$) and for all $b \in B$ edge $\overline{ab}$ meets H. We have

$$\begin{aligned}\eta(H) &\geq \frac{n}{4} \times \frac{3n}{4} \times \frac{1}{2} \\ &= \frac{3n^2}{32}\end{aligned}$$

As a matter of fact the number of the edges intersecting H is exactly $\frac{3n^2}{32}$ in this case.

Case 2: The supporting line ℓ passes through ch(B).

Since the points in B are placed in a small enough circle we have that the line ℓ passes very close to the origin i.e. the euclidean distance between origin and the closest point on ℓ is at most ϵ_r. Among all the edges that are incident on a pair of

[d]Any $\epsilon_r < \frac{2\pi}{3n/4}$ will work but for simiplicity assume ϵ_r is arbitrarily small

vertices in A only a linear (in number of points in A) number of them pass through any circle of randius ϵ_r around origin . That is because for each point a in A there are at most 4 points a' such that the edges of the form $\overline{aa'}$ intersect such a circle due to the small choice of ϵ_r. Since a halfplane passing through q and origin is intersected by $\frac{1}{2} \times \frac{3n}{8} \times \frac{3n}{8}$ edges with both endpoints in A, in this case we have that

$$\begin{aligned}\eta(H) &\geq \frac{3n}{8} \times \frac{3n}{8} \times \frac{1}{2} - cn \\ &= \frac{9n^2}{128} - cn\end{aligned}$$

Above $c \leq 2$.

For the rest of the proof we will assume that the supporting line ℓ does not pass through $ch(B)$. In Case 3 and Case 4 we describe the scenario when ℓ does not intersect $ch(A)$ either. And then for Case 5 and Case 6 we assume that the supporting line ℓ does pass through $ch(A)$.

Case 3: H intersects ch(A) but the supporting line ℓ doesn't.

Note that H can't intersect $ch(B)$ in this case. Let H partition the points in A into two parts A_1 lying above and A_2 lying below H. All edges with one endpoint in A_1 and the other endpoint in A_2 meet H. Similarly all edges with one point in B and other in A_2 lying in positive Y halfspace meet H too. If there are kn points in A_1 then we can assume that $0 \leq k \leq \frac{3}{8}$. We have

$$\begin{aligned}\eta(H) &\geq kn \times (\frac{3}{4} - k)n + (\frac{3}{8} - k)n \times \frac{n}{4} \\ &= \frac{3n^2}{32} + \frac{2n^2k}{4} - n^2k^2 \\ &= \frac{3n^2}{32} + kn(n/2 - kn) \geq \frac{3n^2}{32}\end{aligned}$$

Case 4: H does not intersect ch(A), intersects ch(B) and the supporting line ℓ doesn't intersect ch(B).

Let H partition the points in B into two parts B_1 lying above and B_2 lying below H. All edges with one point in B_1 and other in A lying in the negative Y halfspace meet H. And all the edges with one point in B_2 and other in A lying in positive Y halfspace meet H as well. If there are kn points in B_1 then we can assume that $0 \leq k \leq \frac{1}{8}$. We have

$$\begin{aligned}\eta(H) &\geq kn \times \frac{3n}{8} + (\frac{1}{4} - k)n \times \frac{3n}{8} \\ &= \frac{3n^2}{32}\end{aligned}$$

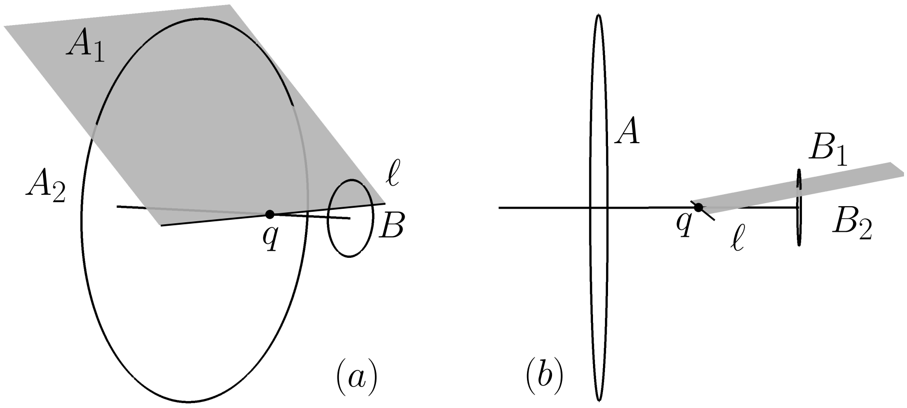

Fig. 6. Figure(a) on the left illustrates Case 3 when H partitions A into A_1 and A_2. Similarly figure(b) on the right illustrates Case 4 when H partitions B into B_1 and B_2.

Case 5: The supporting line ℓ intersects ch(A) but H doesn't intersect ch(B).

H partitions the points in A that lie in positive Y halfspace into two sets A_1 and A_2 such that the points in A_1 and the points in B lie to the left of the plane $\pi(H)$ while the points in A_2 lies to the right.

Let $|A_1| = kn$ and $|A_2| = (3/8 - k)n$. Depending on the value of k we divide this case into two subcases. In each subcase we will show that the constant of n^2 in the value of $\eta(H)$ is large enough i.e., $\eta(H)/n^2 \geq 1/16 + 1/128$.

- $0 \leq k \leq 3/16$: All the edges with one endpoint in $B \cup A_1$ and the other in A_2 meet H. $\eta(H)/n^2 \geq (k + 1/4) \times (3/8 - k)$ which, in the given range, is minimized when $k = 3/16$. In this case we have $\eta(H)/n^2 \geq (1/16 + 5/256)$.
- $3/16 < k \leq 3/8$: Let c and d be the points where $\pi(H)$ meets the circle through the points in the set A and let $\overline{cd}$ be line segment with endpoints c and d. And let e be the point where ℓ meets the XY-plane. Without loss generality assume that $x(c) \geq 0$, $x(d) < 0$, and $x(e) < 0$. Note that e lies on the segment $\overline{cd}$, and the segment $\overline{ce}$ lies in H. When $k > 3/16$ the length of the segment $\overline{ce}$ is more than half of the length of $\overline{cd}$. There are $i \times (3n/4 - i)$ edges with both endpoints in A that intersect $\overline{cd}$ where i is the number of points in A that lie above $\pi(H)$. We know that $kn \leq i \leq 3n/8$. Since the segment $\overline{ce}$ has more than the half of the length of the segment $\overline{cd}$, by symmetry of the points in A more than half of those edges meet H. Also edges with one endpoint in B and other in A_2 intersect H. We have

$$\eta(H)/n^2 \geq 1/2 \times k \times (3/4 - k) + 1/4 \times (3/8 - k)$$

This is minimized, in the given range, when $k = 3/8$. We get $\eta(H)/n^2 \geq 1/16 + 1/128$ in this case.

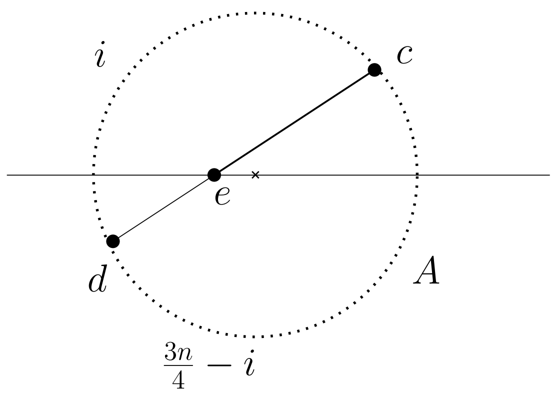

Fig. 7. Case 5: The plane $\pi(H)$ meets the circle through the points in A at c and d. There are $i \times (3n/4 - i)$ edges meeting $\overline{cd}$; at least half of those edges meet $\overline{ec}$ where $n/4 < i < n/2$.

Total number of the edges interesting H is at least $(1/16 + 1/128)n^2$ in both cases.

Case 6: The supporting line ℓ intersects ch(A), H intersects ch(B) but ℓ does not intersect ch(B).

By symmetry of points in A and B we may assume that ℓ intersects the plane $z = 0$ in the negative X halfplane i.e $x(\ell \cap \{p \in \mathbb{R}^3 : z(p) = 0\}) < 0$. Since ℓ passes through the point q at $z = 1/2$, this implies that ℓ intersects the plane $z = 1$ in the positive X halfplane. Also notice that as H intersects B, the direction δ of H can also be assumed to lie in the negative X half of the plane $z = 0$. Let $A' := A \cap \{(x, y, z) : y \geq 0\}$ i.e. the points in A that lie in the positive Y halfspace. H partitions the points in A' into two sets: A_1 lying below H and A_2 lying above. Similarly H partitions B into B_1 lying below H and B_2 lying above. Let $|A_1| = kn$ and $|A_2| = (3/8 - k)n$. Due the small choice of ϵ_r, the radius of circle that contains B, it must be that $|A_1| \leq 2$. Also with $|B_1| = jn$ and $|B_2| = (1/4 - j)n$, it must be that $|B_1| > n/8$. All edges with one endpoint in $A_1 \cup B_1$ and the other endpoint in $A_2 \cup B_2$ meet H. Also all the edges with one endpoint in B_2 and the other endpoint in $A \setminus A'$ meet H. We have

$$\begin{aligned}\eta(H) &\geq jn \times (\frac{3}{8} - k)n + (\frac{3}{8} + k)n \times (\frac{1}{4} - j)n \\ &> (\frac{3}{8} - k)n \times (\frac{n}{4}) \\ &= \frac{3n^2}{32} - \frac{2n}{4}\end{aligned}$$

as required. □

Therefore, unlike the $d = 2$ case where a point of high RS-depth has a correspondingly high Tukey-depth, the three measures in $\mathbb{R}^3$ do not have any such hierarchical relation.

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