Epsilon-Mnets: Hitting Geometric Set Systems with Subsets

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Abstract
The existence of Macbeath regions is a classical theorem in convex geometry [13], with recent applications in discrete and computational geometry. In this paper, we initiate the study of Macbeath regions in a combinatorial setting—and not only for the Lebesgue measure as is the case in the classical theorem—and establish near-optimal bounds for several basic geometric set systems.

1 Introduction
Given a convex body $K$ in $\mathbb{R}^d$ of unit volume, and a parameter $\epsilon > 0$, a classical theorem of Macbeath [13] from convex geometry implies the existence of disjoint convex bodies of $K$, each of volume $\Theta(\epsilon)$, called Macbeath regions, such that any half-space containing at least $\epsilon$-th volume of $K$ completely contains one of these convex bodies. Formally, consider the following theorem (as stated in [6]):

**Theorem A** (Macbeath regions). *Given a convex body $K \subset \mathbb{R}^d$ of unit volume, and a parameter $0 < \epsilon < 1/(2d)^{2d}$, there exists a set $M$ of $O\left(\frac{1}{\epsilon^{1-2d}}\right)$ convex objects such that for any half-space $h$ with $\text{vol}(h \cap K) \geq \epsilon$, there exists a $K_i \in M$ such that $K_i \subset h \cap K$ and $\text{vol}(K_i) \geq \frac{1}{(30d)^d} \cdot \epsilon$.*

Similar partitions of convex bodies was used by Edwald, Larman and Rogers [9] for cap coverings, which were later further extended by Bárány and Larman [5]. They were also used for lower-bounds on range searching by Brönnimann, Chazelle and Pach [6]. Very recently, Macbeath regions were used in an elegant way by Arya, da Fonseca and Mount [3] for computing near-optimal Hausdorff approximations of polytopes. We refer the reader to Bárány [4] for a survey of these and several other applications of Macbeath regions.

Switching over to discrete and combinatorial geometry, a different structure—$\epsilon$-nets—has been developed over the past three decades as a fundamental and powerful tool in computational geometry. Given a set system $(X, \mathcal{R})$, and a parameter $\epsilon$, an $\epsilon$-net is a set $N \subseteq X$ such that $N \cap R \neq \emptyset$ for all $R \in \mathcal{R}$ with $|R| \geq \epsilon |X|$. A famous theorem of Haussler and Welzl [10] states the existence of $\epsilon$-nets of size $O\left(\epsilon^{-d} \log \frac{1}{\epsilon} \right)$ for $(X, \mathcal{R})$, where $d$ is the VC dimension of $\mathcal{R}$. This bound was later improved in [11] to an optimal bound of $(1 + o(1)) \left(\frac{d}{\epsilon} \log \frac{1}{\epsilon}\right)$.

By now $\epsilon$-nets are an indispensable tool in combinatorics, geometry and algorithms (we refer the reader to the books [20, 16, 7, 17] for a small sampling of their constructions and applications).

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The starting point of our work is the observation that the two—$\epsilon$-nets and Macbeath regions—are related. Indeed, theorem 1 implies that for any convex body $K$ in $\mathbb{R}^d$ of volume $V$, it is possible to pick $O(\frac{1}{\epsilon})$ points in $K$ (in fact, even less) which hit all half-spaces containing an $\epsilon$-th fraction of the volume of $K$. However, the statement itself is much stronger than that: instead of just points, it states the existence of $O(\frac{1}{\epsilon})$ regions, each of volume $\Theta(\epsilon V)$, so that any half-space containing an $\epsilon$-th fraction of the volume of $K$ contains one of the regions completely.

As we will prove in this paper, a strengthening of the $\epsilon$-net statement is true for the counting measure for set systems induced by half-spaces in $\mathbb{R}^d$: given any set $P$ of points in $\mathbb{R}^d$, there exist $O(\frac{1}{\epsilon})$ subsets of $P$, each of size $\Theta(\epsilon |P|)$, such that any half-space containing at least $\epsilon \cdot |P|$ points of $P$ contains one of these regions completely. This raises the natural question: of the large number of results known for $\epsilon$-nets for various geometric set systems, which can be optimally strengthened like the case above?

Geometric set systems can be categorized into two frequently studied types. Let $O$ be a family of geometric objects in $\mathbb{R}^d$—e.g., the family of all half-spaces, all balls and so on. We say that $O$ has union complexity $\varphi(\cdot)$ if the combinatorial complexity of the union of any $r$ of the regions of $O$ is at most $r \cdot \varphi(r)$; we refer the reader to the survey [1] for bounds on the union complexity of many geometric objects. Given a set $X$ of points in $\mathbb{R}^d$, we say that $(X, R)$ is a primal set system induced by $O$ if for each $R \in R$, there exists an object $O \in O$ such that $R = X \cap O$. On the other hand, given a finite set $S \subseteq O$ in $\mathbb{R}^d$, we say that $(S, R)$ is a dual set system induced by $S$ if for each $R \in R$, there exists a point $q \in \mathbb{R}^d$ contained in precisely the elements of $R$, i.e., $R = \{O \in S : q \in O\}$.

In this paper we initiate a systematic study of the analogues of Macbeath regions—which we name $\epsilon$-Mnets—for some commonly studied primal and dual geometric set-systems.

**Definition ($\epsilon$-Mnets).** Given a set system $(X, R)$ and a parameter $\epsilon > 0$, a collection $M = \{X_1, \ldots, X_t\}$ of subsets of $X$ is an $\epsilon$-Mnet for $R$ of size $t$ if

1. $|X_i| = \Omega(\epsilon \cdot |X|)$ for each $i = 1, \ldots, t$ and,
2. for every $R \in R$ with $|R| \geq \epsilon \cdot |X|$, there exists an index $j \in \{1, \ldots, t\}$ such that $X_j \subseteq R$.

Furthermore, for any $\kappa \geq 2$, call $M$ a $\frac{1}{\kappa}$-heavy $\epsilon$-Mnet if each set in $M$ has size greater than $\frac{\epsilon |X|}{\kappa}$.

**Our Results**

Our first result establishes tight bounds for the sizes of $\epsilon$-Mnets for the primal and dual set systems induced by axis-parallel rectangles in the plane. This already provides an example where $\epsilon$-Mnets have larger sizes—by factors polynomial in $\frac{1}{\epsilon}$—than $\epsilon$-nets for the corresponding set systems. The proof of the following statement is in Section 2.

**Theorem 1.** Let $\epsilon > 0$, $\kappa \geq 2$ be given parameters.

(a) **Dual set system.** Given a set $S$ of axis-parallel rectangles in the plane, there exist $\frac{1}{\kappa \epsilon}$-heavy $\epsilon$-Mnets of size $O\left(\frac{1}{\epsilon^3} \cdot \frac{1}{\kappa \epsilon^3 + \pi}\right)$ for the dual set-system induced by $S$.

Furthermore, this is near-optimal: for any integer $n > 0$, there exists a set $S$ of $n$ axis-parallel rectangles in $\mathbb{R}^2$ such that any $\frac{1}{\kappa \epsilon}$-heavy $\epsilon$-Mnet for the dual set-system induced by $S$ has size $\Omega\left(\frac{1}{\epsilon^3} \cdot \frac{1}{\kappa \epsilon^3 + \pi}\right)$.

(b) **Primal set system.** Given any set $P$ of points in the plane, there exist $\epsilon$-Mnets of size $O\left(\frac{1}{\epsilon^3} \cdot \frac{1}{\kappa \epsilon^3 + \pi}\right)$ for the primal set-system induced by axis-parallel rectangles on $P$.

Furthermore, this is near-optimal: for any integer $n > 0$, there exists a set $P$ of $n$ points in the plane such that any $\frac{1}{\kappa \epsilon}$-heavy $\epsilon$-Mnet for the primal set-system induced by axis-parallel rectangles on $P$ has size $\Omega\left(\frac{1}{\epsilon^3} \cdot \frac{1}{\kappa \epsilon^3 + \pi}\right)$.  

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Our next result states the existence of small $\epsilon$-Mnets for dual set systems as a function of the union complexity of the objects. Call a set $S$ of objects in $\mathbb{R}^d$ well-behaved if for any subset $S' \subseteq S$ and any $Q \subseteq \mathbb{R}^d$, one can decompose the cells in the arrangement of $S'$ that intersect $Q$ into cells of constant descriptive complexity, where the complexity of this decomposition is proportional to the total number of vertices in the cells that intersect $Q$; we refer the reader to [8] for more details. The proof of the following statement is in Section 3.

**Theorem 2.** Let $\mathcal{R}$ be the dual set system induced by a set of well-behaved regions $S$ in $\mathbb{R}^d$ with union complexity $\varphi(\cdot)$ and let $\epsilon > 0$ be a given parameter. Then there exists an $\epsilon$-Mnet for $\mathcal{R}$ of size $O\left(\frac{1}{\epsilon^\varphi(\frac{1}{\epsilon})}\right)$.

Interestingly, as $\varphi(m) = \Omega(m)$ for the dual set system induced by axis-parallel rectangles in the plane, Theorem 1 implies that the dependence of $\varphi(\cdot)$ in Theorem 2 cannot be reduced to, for example, $\log \varphi(\cdot)$, as is the case for $\epsilon$-nets.

Our last result is to consider the primal case where the input is a set of points and the set system is defined by containment by geometric objects such as disks, lines, triangles and more generally, $k$-sided polygons in the plane. The proof of the following statement is in Section 4.

**Theorem 3.** Let $P$ be a set of $n$ points, and $\epsilon > 0$ a given parameter. Then one can construct $\epsilon$-Mnets of size:

(a) $O\left(\frac{1}{\epsilon^{\lfloor d/2 \rfloor}}\right)$ for the primal set system induced by half-spaces in $\mathbb{R}^d$, for $d \geq 2$.

Furthermore, this cannot be improved substantially: for any integers $d \geq 2$ and $n > 0$, there exists a set of $n$ points in $\mathbb{R}^d$ such that any $\epsilon$-Mnet for the primal set system induced by half-spaces has size $\Omega\left(\frac{1}{\epsilon^{\lfloor d/2 \rfloor}}\right)$.

(b) $O\left(\frac{1}{\epsilon}\right)$ for the primal set system induced by disks in the plane.

(c) $O\left(\frac{1}{\epsilon^3} (\log \frac{1}{\epsilon})^4\right)$ for the primal set system induced by triangles, and in general $k$-sided polygons in the plane (the constant in the asymptotic notation depends on $k$).

(d) $O\left(\frac{1}{\epsilon^2} (\log \frac{1}{\epsilon})^2\right)$ for the primal set system induced by lines, $O\left(\frac{1}{\epsilon^2} (\log \frac{1}{\epsilon})^3\right)$ for the one induced by cones, and $O\left(\frac{1}{\epsilon^2} (\log \frac{1}{\epsilon})^4\right)$ for the one induced by strips in the plane.

Furthermore, this is near-optimal: for any integer $n > 0$, there exists a set of $n$ points in $\mathbb{R}^2$ such that any $\epsilon$-Mnet for the primal set system induced by lines or cones or strips has size $\Omega\left(\frac{1}{\epsilon^2}\right)$.

(e) $O\left(\frac{1}{\epsilon}\right)$ for the primal set system induced by axis-parallel rectangles in $\mathbb{R}^2$, all intersecting the $y$-axis.

Theorem 3 implies that near-linear bounds for $\epsilon$-Mnets are not possible for even simple primal set-systems such as those induced by lines in the plane. This contrasts sharply with $\epsilon$-net bounds for geometric set systems, which are near-linear for any set system with constant VC dimension.

## 2 Proof of Theorem 1

The following lemma, of independent interest, gives insight for studying $\epsilon$-Mnets for both the primal and dual set systems induced by axis-parallel rectangles in the plane.

**Lemma 2.1.** For any integers $r, d \geq 3$, consider the grid $G = \{0, \ldots, r - 1\}^d$ in $\mathbb{R}^d$ consisting of $r^d$ points. Then there exists a bijective mapping $\pi : G \mapsto \mathbb{R}^2$ such that the primal set system on $G$ induced by axis-parallel lines can be realized by the primal set system induced by axis-parallel rectangles in $\mathbb{R}^2$ on the set $\{\pi(p), p \in G\}$. 

Proof. Let $[r]$ represent the set $\{0, \cdots, r-1\}$. For any $i \in \{1, \cdots, d\}$ and integers $a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_d \in [r]$, consider the set of points

$$S_i(a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_d) = \{(a_1, \cdots, a_{i-1}, t, a_{i+1}, \cdots, a_d) : t \in [r]\}.$$  

We call such a set a line in direction $i$. There are $dr^{d-1}$ such lines, $r^{d-1}$ in each of the $d$ directions (along the axes) in $\mathbb{R}^d$.

We will show that there exists a mapping $\pi : G \mapsto \mathbb{R}^2$ such that for each line $l$ in any direction, the inclusion-minimal axis-parallel rectangle containing the image, under $\pi(\cdot)$, of the points in $l$ does not contain the image of any other point of $G$. Here is the mapping $\pi(\cdot)$ that we will use:

$$\pi((a_1, \cdots, a_d)) = \sum_j a_j \vec{v}_j, \quad \text{where } \vec{v}_j = (r^j, r^{d+1-j}).$$

For any point $z \in G$, we will interpret $p = \pi(z)$ both as a vector and as a point, as suitable. When treating it as a vector, we will denote it by $\vec{p}$. For any $z' = (a_1, \cdots, a_d) \in G$, let $\vec{V}_{<i}(z')$ denote the vector $\sum_{j<i} a_j \vec{v}_j$ and $\vec{V}_{\geq i}(z')$ denote the vector $\sum_{j\geq i} a_j \vec{v}_j$. Thus we can write $\pi(z') = \vec{V}_{<i}(z') + a_i \vec{v}_i + \vec{V}_{\geq i}(z')$.

Consider any line, say $l = S_i(a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_d)$, and let $R$ be the smallest rectangle containing the set of $r$ mapped points of $l$ in the plane, namely the set

$$f(l) = \{\pi((a_1, \cdots, a_{i-1}, t, a_{i+1}, \cdots, a_d)) : t \in [r]\}.$$

Let $z_l = (a_1, \cdots, a_{i-1}, 0, a_{i+1}, \cdots, a_d)$ and $z_r = (a_1, \cdots, a_{i-1}, r-1, a_{i+1}, \cdots, a_d)$ be the two extreme points lying on $l$. As all the coordinates except the $i$-th one are the same for all points lying on $l$, the mapped point with the maximum $x$-coordinate is the one that maximizes $t \cdot r^i$, i.e., the point $\pi(z_r)$. Similarly, $\pi(z_l)$ has the maximum $y$-coordinate, and $\pi(z_l)$ has the minimum $x$- and $y$-coordinates. Furthermore, the width of $R$ is defined by the difference in the $x$-coordinates of $\pi(z_r)$ and $\pi(z_l)$, and so it is precisely $(r-1)r^i$. Likewise, the height of $R$ is $(r-1)r^{d+1-i}$.

It remains to show that for any other point, say $z = (b_1, \cdots, b_d) \in G \setminus l$, $\pi(z)$ does not lie in $R$. Let $z' = (a_1, \cdots, a_{i-1}, b_i, a_{i+1}, \cdots, a_d) \in G$ be the point lying on the line $l$ with the same $i$-th coordinate as $z$. Let $p = \pi(z) = \vec{V}_{<i}(z) + b_i \vec{v}_i + \vec{V}_{\geq i}(z)$ and $q = \pi(z') = \vec{V}_{<i}(z') + b_i \vec{v}_i + \vec{V}_{\geq i}(z')$. Then

$$\vec{p} - \vec{q} = (\vec{V}_{<i}(z) - \vec{V}_{<i}(z')) + (\vec{V}_{\geq i}(z) - \vec{V}_{\geq i}(z')).$$

Since $\vec{p} \neq \vec{q}$, one of the above two summands must be non-zero. Without loss of generality assume that the second summand is non-zero. The other case is similar. As $\vec{V}_{\geq i}(z) - \vec{V}_{\geq i}(z') = \sum_{j>i} (b_j - a_j) \vec{v}_j$, it is a non-zero integral combination of the vectors $\vec{v}_j$ for $j > i$, and so its $x$-coordinate has magnitude at least $r^{i+1}$. On the other hand the $x$-coordinate of $(\vec{V}_{<i}(z) - \vec{V}_{<i}(z'))$ has magnitude at most $\sum_{1 \leq j < i} (r-1)j = r^i - r$. Therefore the difference in the $x$-coordinates between $p$ and $q$ is at least $r^{i+1} - (r^i - r)$, which is greater than the width of $R$. Hence, $p \notin R$. When $(\vec{V}_{<i}(z) - \vec{V}_{<i}(z')) \neq 0$, a similar argument holds for the $y$-coordinates of $p$ and $q$, showing that the difference in their $y$-coordinates is larger than the height of $R$. \qed

**Case (a): Dual set system.**

**Lower-bound.** We now show that for any integers $\kappa \geq 2$ and $n \geq 0$, there exists a set $\mathcal{R}$ of $n$ axis-parallel rectangles such that any $\frac{1}{\kappa}$-heavy $\epsilon$-Mnet for the dual set system induced by $\mathcal{R}$ has size $\Omega(\frac{1}{\kappa+1}(n-\epsilon))$. Apply Lemma 2.1 with $d = \kappa$ and $r = \epsilon^{-\frac{1}{\kappa+1}}$. Let $G$ be the grid $[r]^d$ as before. We set $P = \{\pi(p) : p \in G\}$ and let $\mathcal{R}'$ be the set of $dr^{d-1}$ rectangles corresponding to the $dr^{d-1}$ lines in $G$. Construct the required set $\mathcal{R}$ by
replacing each rectangle of \( \mathcal{R}' \) with \( \frac{\epsilon k}{\epsilon} \) copies. Note that \( |\mathcal{R}| = \frac{n}{k} \cdot d r^{d-1} = n \). Since each of the points in \( G \) is contained in \( d \) lines (one in each direction), each point of \( P \) is contained in \( d \) rectangles of \( \mathcal{R}' \) and consequently \( \epsilon n \) rectangles of \( \mathcal{R} \). Since there is at most one line through two points in \( G \), there are at most \( \frac{n}{\epsilon} \) rectangles of \( \mathcal{R} \) that contain any pair of points \( p, q \in P \). Since for any \( \frac{1}{\epsilon} \)-heavy \( \epsilon \)-Mnet \( \mathcal{M} \), each \( U \in \mathcal{M} \) has size greater than \( \frac{\epsilon n}{\epsilon} \), it must be that no set in \( \mathcal{M} \) can be contained in two sets \( \mathcal{R}(p) \) and \( \mathcal{R}(q) \) induced by two distinct points \( p \) and \( q \) in \( P \). Therefore \( |\mathcal{M}| \geq |P| = r^d = \epsilon^{-\frac{n}{\epsilon}} = \frac{1}{\epsilon^{1+\frac{1}{d}}}. \)

Upper-bound. We now establish an upper-bound for the dual set systems induced by axis-parallel rectangles in the plane.

Construct a hierarchical subdivision on \( S \), as follows. Let \( k = \left\lceil \frac{1}{\epsilon^{1/n}} \right\rceil \), and for \( i = 0, \ldots, \kappa \), set the parameters \( n_i = \frac{n}{k^i} \), and \( \epsilon_i = \epsilon \left( \frac{k}{2} \right)^i \). At the 0-th level (here \( i = 0 \)), let \( l_0^0, \ldots, l_{k-1}^0 \) be a set of \( k-1 \) vertical lines such that the number of rectangles of \( S \) lying between two consecutive lines—call this region a ‘slab’—is at most \( \frac{n}{k} \). Let \( S^0_j \) be the set of rectangles lying entirely in the \( j \)-th slab. For each index \( j = 1, \ldots, k-1 \), construct a \( \frac{\epsilon}{\epsilon_i} \)-Mnet for all the rectangles of \( S \) intersecting \( l_j^0 \). Furthermore, construct an \( \epsilon \left( \frac{k}{2} \right) \)-Mnet for the rectangles in \( S^0_j \), for each \( j = 1, \ldots, k-1 \) in the similar manner as above. The construction continues for \( \kappa \) steps: at the \( i \)-level, there are \( k^i \) total sub-problems, each sub-problem consists of at most \( n_i = \frac{n}{k^i} \) rectangles and with \( \epsilon_i = \epsilon \left( \frac{k}{2} \right)^i \).

At the base case of the recursion, we use a direct \( O \left( \frac{1}{\epsilon^2} \right) \)-sized construction for the \( \epsilon \)-Mnet of the \( k^n \) sub-problems at the last \( \kappa \)-level: for the sub-problem of computing a \( \epsilon \)-Mnet for a set of rectangles \( S' \) where \( |S'| \leq n_{\kappa} \), construct a set \( L' \) of \( \frac{\kappa}{\epsilon_{\kappa}} \) vertical and \( \frac{\kappa}{\epsilon_{\kappa}} \) horizontal lines such that each vertical (resp. horizontal) slab induced by \( L' \) contains at most \( \frac{\epsilon_{\kappa}|S'|}{\epsilon} \) vertical (resp. horizontal) boundary edges of the rectangles in \( S' \). For each bounded cell \( c \) induced by \( L' \), add to \( \mathcal{M} \) all the rectangles of \( S' \) completely containing \( c \), if their total number is at least \( \frac{\epsilon_{\kappa}|S'|}{2} \) (Fig. 5). Now take any point \( q \in \mathbb{R}^2 \) lying in at least \( \epsilon_{\kappa}|S'| \) rectangles of \( S' \) and let \( c \) be the cell induced by \( L' \) containing \( q \). At least one of the boundary edges of any rectangle \( R \) containing \( q \) but not containing \( c \) must lie in the vertical or horizontal slab induced by \( L' \) containing \( q \). Thus there can be only \( \epsilon_{\kappa}|S'| \) such rectangles that contain \( q \) but not the cell \( c \). The remaining at least \( \frac{\epsilon_{\kappa}|S'|}{2} \) rectangles that contain \( q \) must then all contain \( c \), and so would form a set in \( \mathcal{M} \) of size at least \( \frac{\epsilon_{\kappa}|S'|}{2} \). Note that the total number of sets added to \( \mathcal{M} \) is \( O \left( \frac{1}{\epsilon_{\kappa}} \right) \).

The next two claims conclude the proof by showing that all these Mnets together form an \( \epsilon \)-Mnet \( \mathcal{M} \) for \( S \) of the required size.

**Claim 1.** Each set in \( \mathcal{M} \) has size \( \Theta \left( \frac{cn}{\epsilon^2} \right) \). The size of \( \mathcal{M} \) is \( O \left( \frac{4^n}{\epsilon^{1+\frac{1}{d}}} \right) \).

**Proof.** At the \( i \)-level there are \( k^i \) sub-problems, each of size at most \( n_i = \frac{n}{k^i} \) with \( \epsilon_i = \epsilon \left( \frac{k}{2} \right)^i \). For each such sub-problem, we partition its set of at most \( n_i \) rectangles by \( k-1 \) lines, and construct a \( \frac{\epsilon}{\epsilon_i} \)-Mnet for the rectangles intersecting these \( k-1 \) lines. Note that the set of rectangles intersecting any line, and clipped to one side of the line have linear union complexity [1] and by Theorem 2 there exists a \( \frac{\epsilon}{\epsilon_i} \)-Mnet of size \( O \left( \frac{1}{\epsilon_i} \right) \). Hence the total size over all internal sub-problems is:

\[
\sum_{i=0}^{\kappa} k^i \cdot (k-1) \cdot O \left( \frac{1}{\epsilon_i} \right) \leq \sum_{i=0}^{\kappa} k^{i+1} \cdot O \left( \frac{2^i}{\epsilon k^i} \right) = \sum_{i=0}^{\kappa} O \left( \frac{2^i}{\epsilon^{1+\frac{1}{d}}} \right) = O \left( \frac{2^n}{\epsilon^{1+\frac{1}{d}}} \right).
\]

At the last level, after \( \kappa \) steps, we have \( k^n \) sub-problems, each with at most \( \frac{n}{k^n} \) rectangles, and \( \epsilon_{\kappa} = \epsilon \left( \frac{k}{2} \right)^{\kappa} \). Now use a direct construction which constructs an \( \epsilon \)-Mnet of size \( O \left( \frac{1}{\epsilon_i} \right) \), to get the total size of Mnet at the last step to be \( O(\kappa) \cdot \frac{1}{\epsilon_{\kappa}} = O \left( \frac{4^n}{\epsilon^{2+\frac{1}{d}}} \right) = O \left( \frac{4^n}{\epsilon} \right) \).

At any level \( i \), we construct a \( \epsilon \)-Mnet on a set of at most \( \frac{n}{k^i} \) rectangles. So each set in the constructed Mnet has size \( \Omega \left( \epsilon_i \cdot \frac{n}{k^i} \right) = \Omega \left( \frac{cn}{\epsilon^2} \right) = \Omega \left( \frac{cn}{\epsilon^2} \right) \).
Claim 2. For each point \( q \in \mathbb{R}^2 \) lying in at least \( \epsilon n \) rectangles of \( S \), there exists a set \( U \in \mathcal{M} \) such that \( q \) lies in all the rectangles of \( U \).

Proof. Take a point \( q \) lying in at least \( \epsilon n \) rectangles of \( S \). At the 0-th level, say \( q \) lies in the vertical slab defined by lines \( l_j^0 \) and \( l_{j+1}^0 \). If \( q \) is contained in at least \( \frac{\epsilon n}{2} \) rectangles intersected by either \( l_j^0 \) or \( l_{j+1}^0 \), say \( l_j^0 \), then it is contained in at least \( \frac{\epsilon n}{4} \) rectangles out of a total of at most \( n \) rectangles intersected by \( l_j^0 \). So the \( \frac{\epsilon}{4} \)-Mnet for \( l_j^0 \) will have a set \( U \) such that each rectangle in \( U \) contains \( q \). Otherwise \( q \) is contained in at least \( \frac{\epsilon n}{2} = \epsilon \left( \frac{k}{2} \right) \cdot \frac{n}{k} = \epsilon i n_i \) rectangles of the set \( S_j \) of size at most \( n_1 = \frac{n}{k} \), and we proceed to this sub-problem.

In general, at the \( i \)-level, each sub-problem has at most \( n_i = \frac{n}{k^i} \) rectangles, with \( \epsilon_i = \epsilon \left( \frac{k}{2} \right)^i \). Then either \( q \) is contained in at least \( \epsilon n_i \) rectangles intersecting one of the lines, and so will contain a set from the \( \frac{\epsilon}{4} \)-Mnet constructed for each of the \( k - 1 \) vertical lines. Or \( q \) is contained in at least \( \frac{\epsilon n_i}{2} \) rectangles out of a total of at most \( n_{i+1} = \frac{n}{k} \) rectangles lying in one of the slabs defined by the \( k - 1 \) vertical lines. But as

\[
\frac{\epsilon_i n_i}{2} = \epsilon \cdot \left( \frac{k}{2} \right)^i \cdot \frac{n}{k^i} = \epsilon \left( \frac{k}{2} \right)^{i+1} \cdot \frac{n}{k^{i+1}} = \epsilon (i+1)n_{i+1},
\]

\( q \) will be covered inductively by the \( \epsilon(i+1) \)-Mnet constructed for the \( n_{i+1} \) rectangles in one of the resulting sub-problems at level \( i + 1 \).

\[\] 

Case (b): Primal set system.

Lower-bound. We now show that for any integers \( \kappa \geq 2 \) and \( n \geq 0 \), there exists a set \( P \) of \( n \) points in \( \mathbb{R}^2 \) such that any \( \frac{1}{\kappa} \)-heavy \( \epsilon \)-Mnet of \( P \) for the primal set system induced by axis-parallel rectangles in the plane has size \( \Omega \left( \frac{1}{\kappa} \log_{\frac{1}{\kappa}} \frac{\epsilon}{2} \right) \). Apply Lemma 2.4 with \( r = \kappa \), and with parameter \( d \) set with \( \epsilon^{d-1} = \frac{1}{2} \). According to the lemma, there is a mapping \( \pi \) from the grid \( G = \{0, \ldots, r - 1\}^d \) to the plane so that for each subset \( S \subset G \) of the grid obtained by intersecting \( G \) with an axis-parallel line, there exists an axis-parallel rectangle \( R \) in the plane such that \( R \cap \pi(G) = \pi(S) \); i.e., \( R \) contains exactly the mapped points of \( S \). There are \( d r^{d-1} = \Theta \left( \frac{1}{\kappa} \log_{\frac{1}{\kappa}} \frac{\epsilon}{2} \right) \) such subsets and let \( R \) be the set of axis-parallel rectangles corresponding to these. Let \( P \) be the set of points obtained by replacing each point \( p \in \pi(G) \) with \( \frac{n}{r} \) copies of \( p \) (note that \( P \) is not a multi-set; think of each copy of the same point in \( \pi(G) \) as a distinct point). The number of points in \( P \) is \( r^{d-1} \cdot \frac{n}{r} = n \). Each rectangle in \( R \) contains \( r \cdot \frac{n}{r} = \epsilon n \) points of \( P \). Also, any pair of rectangles in \( R \) share at most \( \frac{n}{r} = \epsilon \frac{n}{2} \) points of \( P \). Thus no two rectangles in \( R \) may share the same set \( U \in \mathcal{M} \) of a \( \frac{1}{\kappa} \)-heavy \( \epsilon \)-Mnet \( M \). Since each of them must contain some \( U \in \mathcal{M} \), we have \( |\mathcal{M}| \geq |R| \) and the result follows.

Upper-bound. We now present a matching upper-bound for the primal set system induced by axis-parallel rectangles in the plane.

Assume \( P = \{p_1, \ldots, p_n\} \) are labeled in the order of increasing \( x \)-coordinates. Given \( P \), construct a balanced binary subdivision of \( P \) with vertical lines: divide \( P \) by a vertical line into two equal-sized subsets \( P^1_0, P^1_1 \), and then recursively divide each of these sets into two equal-sized subsets and so on for \( \log \frac{1}{\epsilon} \) levels. At the \( i \)-th level of recursion, there are \( 2^i \) sets of size \( \frac{n}{2^i} \).

Let \( P^i_j \) denote the \( j \)-th subset of \( P \) at level \( i \), i.e.,

\[
\text{For } 1 \leq i \leq \log \frac{1}{\epsilon}, \quad 0 \leq j < 2^i, \quad P^i_j = \{p_j, p_{j+1}, \ldots, p_{(j+1)2^i}\}.
\]

For each set \( P^i_j \), and for each of its two bounding lines, say lines \( l_0 \) and \( l_1 \), construct a \( 2 \cdot \epsilon \)-Mnet for the following primal set-system: the base set is \( P^i_j \), and given a line \( l \in \{l_0, l_1\} \), the sets are induced by axis-parallel rectangles intersecting the line \( l \). Note that all points of \( P^i_j \) lie on the same side of \( l \). Let \( M \) be the union of all
these Mnets. Crucially, the primal set system induced by the set of axis-parallel rectangles on the same side of \( l \) admits an \( \epsilon \)-Mnet of size \( O(\frac{1}{\epsilon}) \) by Theorem 3(e).

We now prove that \( \mathcal{M} \) is an \( \epsilon \)-Mnet of \( P \), of size \( O(\frac{1}{\epsilon} \log \frac{1}{\epsilon}) \).

**Claim 3.** Each set in \( \mathcal{M} \) has size \( \Theta(\epsilon n) \), and size of \( \mathcal{M} \) is \( O(\frac{1}{\epsilon} \log \frac{1}{\epsilon}) \).

**Proof.** The set \( P_j^i \) has size \( \frac{n}{2^i} \), and so each set in a \((2^{i-1}\epsilon)\)-Mnet of \( P_j^i \) has size \( \Omega(2^{i-1}\epsilon \cdot \frac{n}{2^i}) = \Omega(\epsilon n) \). Note that each \( 2^{i-1}\epsilon \)-Mnet has size \( O(\frac{1}{2\epsilon^i}) \), there are \( 2^i \) sets \( P_j^i \) at level \( i \), and a total of \( \log \frac{1}{\epsilon} \) levels. Hence the size of \( \mathcal{M} \) is \( O(\frac{1}{2\epsilon^i} \cdot 2^i \cdot \log \frac{1}{\epsilon}) = O(\frac{1}{\epsilon} \log \frac{1}{\epsilon}) \). \( \blacksquare \)

**Claim 4.** Each axis-parallel rectangle containing at least \( \epsilon n \) points of \( P \) contains a set of \( \mathcal{M} \).

**Proof.** Let \( R \) be an axis-parallel rectangle containing at least \( \epsilon n \) points of \( P \). Let \( i \) be the smallest index such that \( R \) intersects exactly one vertical line separating two sets \( P_j^i \) and \( P_{j+1}^i \) at level \( i \). Say \( R \) intersects the line \( l \) separating \( P_j^i \) and \( P_{j+1}^i \). Then \( R \) must contain at least \( \frac{n}{2^i} \) points from either \( P_j^i \) or \( P_{j+1}^i \), say \( P_j^i \). Let \( R' \) be the part of \( R \) on the side of \( l \) towards \( P_j^i \). Thus \( R' \) must contain at least one set of the \( 2^{i-1}\epsilon \)-Mnet for \( P_j^i \), as

\[
|R \cap P_j^i| = |R' \cap P_j^i| \geq \frac{\epsilon n}{2} = 2^{i-1}\epsilon \cdot \frac{n}{2^i} = 2^{i-1}\epsilon \cdot |P_j^i|.
\]

\( \blacksquare \)

### 3 Proof of Theorem 2

Given the input set \( S \) of regions in \( \mathbb{R}^d \), define the *depth* of any point \( q \in \mathbb{R}^d \) with respect to \( S \) to be the number of regions of \( S \) containing \( q \). The key tool used in the proof are *shallow cuttings*:

**Theorem B ([15, 8]).** Given a set \( S \) of \( n \) well-behaved regions in \( \mathbb{R}^d \) with union complexity \( \varphi(\cdot) \) and two parameters \( r, l > 0 \), there exists a partition of \( \mathbb{R}^d \) into a set \( \Xi \) of interior-disjoint cells (of constant description complexity) such that

1. each cell of \( \Xi \) is intersected by the boundary of at most \( \frac{n}{r} \) regions of \( S \), and
2. the number of cells in \( \Xi \) that contain points of depth less than \( l \) (with respect to \( S \)) is \( O\left(\left(\frac{n}{r}+1\right)^d \cdot \frac{n}{l} \cdot \varphi\left(\frac{n}{l}\right)\right) \).

Such a partition \( \Xi \) is called a \( (\frac{1}{r}, l) \)-shallow cutting of \( S \).

We will construct the required \( \epsilon \)-Mnet \( \mathcal{M} \) as a union of \( \log \frac{1}{\epsilon} \) collections \( \mathcal{M}_i \), for \( i = 0, \ldots, \log \frac{1}{\epsilon} \). For a fixed index \( i \), construct the sets in \( \mathcal{M}_i \) by setting \( l_i = 2^{i+1}\epsilon n \), \( r_i = \frac{1}{2^{i+1}\epsilon} \), and construct a \((\frac{1}{r_i}, l_i)\)-shallow cutting, denoted by \( \Xi_i \), for \( S \). Call a cell \( \Delta \in \Xi_i \) *shallow* if it contains points of depth less than \( l_i \). For each \( \Delta \in \Xi_i \), let \( r(\Delta) \) be the set of regions in \( S \) that completely contain \( \Delta \); i.e., \( S \in r(\Delta) \) if and only if \( \Delta \subseteq S \). Now, for all shallow cells \( \Delta \) with \( r(\Delta) \geq \frac{n}{r} \), add \( r(\Delta) \) to \( \mathcal{M}_i \).

We can trivially upper-bound \( |\mathcal{M}_i| \) by the number of shallow cells of \( \Xi_i \), i.e., cells containing a point of depth less than \( l_i = 2^{i+1}\epsilon n \). Thus using Theorem B we get

\[
|\mathcal{M}_i| = O\left(\left(\frac{r_i \cdot 2^{i+1}\epsilon n}{n} + 1\right)^d \cdot \frac{n}{2^{i+1}\epsilon n} \cdot \varphi\left(\frac{n}{2^{i+1}\epsilon n}\right)\right) = O\left(4^d \cdot \frac{1}{2\epsilon} \cdot \varphi\left(\frac{1}{2\epsilon}\right)\right).
\]
First we bound the size of $M = \bigcup_i M_i$:

$$|M| \leq \sum_{i=0}^{\log \frac{1}{\epsilon}} |M_i| = \sum_{i=0}^{\log \frac{1}{\epsilon}} O\left(4^d \cdot \frac{1}{2^i} \cdot \varphi\left(\frac{1}{2^i}\right)\right) = O\left(4^d \cdot \frac{1}{\epsilon} \cdot \varphi\left(\frac{1}{\epsilon}\right)\right) \sum_{i=0}^{\log \frac{1}{\epsilon}} \frac{1}{2^i} = O\left(4^d \cdot \frac{1}{\epsilon} \cdot \varphi\left(\frac{1}{\epsilon}\right)\right).$$

To see that sets in $M$ form the required $\epsilon$-Mnet, let $p \in \mathbb{R}^d$ be any point contained in $t$ regions of $S$, where $t \geq \epsilon n$. Let $i$ be the index such that $2^i \epsilon n \leq t < 2^{i+1} \epsilon n$. Let $\Delta_p$ be the shallow cell in the $(\frac{1}{2^i}, l_i)$-shallow cutting that contains $p$. Recall that the $(\frac{1}{2^i}, l_i)$-shallow cutting $\Xi_i$ partitions $\mathbb{R}^d$ into a set of cells such that each cell intersects the boundary of at most $\frac{2^i}{\epsilon n} = 2^{i-1} \epsilon n$ objects in $S$. Thus, of all the $t \geq 2^i \epsilon n$ regions containing $p$, the boundary of at most $2^i \epsilon n$ regions can intersect $\Delta_p$. The remaining at least $2^{i-1} \epsilon n \geq \frac{\epsilon n}{2}$ regions of $S$ containing $p$ must then completely contain $\Delta_p$, and so are in the set $r(\Delta_p)$. Thus the set $r(\Delta_p)$ is added to $M_i$, and we have $|r_i(\Delta)| \geq 2^{i-1} \epsilon n \geq \frac{\epsilon n}{2}$.

### 4 Proof of Theorem 3

(a). First we establish the upper-bound on the sizes of $\epsilon$-Mnets for the primal set system induced by half-spaces in $\mathbb{R}^d$. For a point $p \in \mathbb{P}$, let $H_p$ be its dual hyperplane, and let $\mathbb{H} = \{H_p \mid p \in \mathbb{P}\}$. Let $\mathbb{H}^+$ (resp. $\mathbb{H}^-$) be a set of upward-facing (resp. downward-facing) half-spaces defined by $\mathbb{H}$. Apply Theorem 2 to the dual set system induced by $\mathbb{H}^+$ (resp. $\mathbb{H}^-$) to get an $\epsilon$-Mnet $M^+$ (resp. $M^-$), and let $M$ be the corresponding collection of sets of $\mathbb{P}$ corresponding to both $M^+$ and $M^-$. As $M^+$ (resp. $M^-$) is an $\epsilon$-Mnet for $\mathbb{H}^+$ (resp. $\mathbb{H}^-$), for any point $q \in \mathbb{R}^d$ contained in at least $\epsilon n$ half-spaces in $\mathbb{H}^+$ (resp. $\mathbb{H}^-$), there exists a set in $M^+$ (resp. $M^-$) of size $\Omega(\epsilon n)$, such that each half-space in this set contains $q$. Switching to the primal viewpoint, any upward-facing (resp. downward-facing) half-space $H_q$ containing at least $\epsilon n$ points of $\mathbb{P}$, corresponds in the dual to a point $q$ that is contained in at least $\epsilon n$ downward-facing (resp. upward-facing) half-spaces in $\mathbb{H}^+$ (resp. $\mathbb{H}^-$). As $M^+$ (resp. $M^-$) is an $\epsilon$-Mnet for $\mathbb{H}^+$ (resp. $\mathbb{H}^-$), it follows that $M$ is an $\epsilon$-Mnet for the primal set system induced by half-spaces. To bound the size of $M$ obtained from Theorem 2, it suffices to note that for half-spaces, $r(\varphi(r) = O(r[d/2])$.

For the lower-bound for $\epsilon$-Mnets for the primal set system induced by half-spaces in $\mathbb{R}^d$, we first prove the following more general theorem.

**Theorem 4.** Given a real parameter $\epsilon > 0$, integer $n > 1$ and two constants $\delta$ and $k$, there exists a set $\mathbb{P}$ of $n$ points in the plane, and a set $\mathbb{D} = \Omega(\frac{1}{\epsilon^{d+1}})$ curves, each of degree at most $\delta$, such that a) each curve contains $\epsilon n$ points of $\mathbb{P}$ and b) no two curves in $\mathbb{D}$ have more than $\frac{\delta k}{\epsilon}$ points in common. In particular, any $\frac{1}{\epsilon}$-heavy $\epsilon$-Mnet for the primal set system on $\mathbb{P}$ induced by curves of degree at most $\delta$ has size $\Omega(\frac{1}{\epsilon^{d+1}})$ (the constants in the asymptotic notation depend on $k$ and $\delta$).

**Proof.** Denote by $G$ the set of $\frac{\delta k}{\epsilon}$ grid points in $\{0, \ldots, \delta k - 1\} \times \{0, \ldots, \left\lceil \frac{1}{\epsilon} \right\rceil - 1\}$. The set of curves in $\mathbb{D}$ will be all univariate functions in $x$ of the form

$$y = \delta \sum_{i=0}^{\delta} a_i \cdot x^i,$$

where each $a_i \in \{0, 1, \ldots, \left\lfloor \frac{1}{\epsilon} \right\rfloor - 1\}$.

Clearly we have

$$|\mathbb{D}| = \prod_{i=0}^{\delta} \frac{1}{\epsilon^{(\delta + 1)(\delta k)^i}} = \Omega\left(\frac{1}{\epsilon^{\delta + 1} (\delta k)^{\Theta(2^i)}}\right) = \Omega\left(\frac{1}{\epsilon^{\delta + 1}}\right).$$
Since for each value of $x \in \{0, \ldots, \delta k - 1\}$, the corresponding value of $y$ for each of the curves in $D$ lies in $\{0, \ldots, \left\lceil \frac{1}{e} \right\rceil - 1\}$, each of the curves of $D$ contain precisely $\delta k$ points of $G$. Furthermore, as these curves have degree at most $\delta$, no two intersect in more than $\delta$ points of $G$.

Let $P$ be the set of $n$ points obtained by replacing each point of $G$ with $\frac{\epsilon n}{\delta k}$ copies to get a set of $n$ points in the plane. Now each curve in $D$ contains $\delta k \cdot \frac{\epsilon n}{\delta k} = \epsilon n$ points of $P$ and every pair of curves have less than $d \cdot \frac{\epsilon n}{\delta k} = \epsilon n$ points of $P$ in common.

Finally observe that any $\frac{1}{k}$-heavy $\epsilon$-Mnet $M$ for the primal set system on $P$ induced by $D$ must consist of at least $|D|$ sets: each curve $D \in D$ must completely contain a set $R \in M$ of size at least $\frac{\epsilon n}{k}$, and furthermore $R$ cannot be contained in any other curve $D' \in D$, as any two curves of $D$ have less than $\frac{\epsilon n}{k}$ points of $P$ in common. \(\square\)

Now we show the desired lower-bound for $\epsilon$-Mnets for the primal set system induced by half-spaces in $\mathbb{R}^d$.

**Corollary 4.1.** For any $\epsilon > 0$ and integers $n$ and $d$, there exists a set $P$ of $n$ points in $\mathbb{R}^d$ such that any $\epsilon$-Mnet for the primal set system on $P$ induced by half-spaces has size $\Omega\left(\frac{1}{\epsilon \left\lfloor \frac{d-2}{3} \right\rfloor}\right)$.

**Proof.** First assume that $\frac{d-2}{3}$ is an integer, and apply Theorem 4.1 with $\delta = \frac{d-2}{3}$ and $k = 2$ to get a set $P$ of $n$ points in $\mathbb{R}^2$ and a set $D$ of curves such that any $\epsilon$-Mnet for the primal set system induced by $D$ on $P$ has size $\Omega\left(\frac{1}{\epsilon \left\lfloor \frac{d-2}{3} \right\rfloor}\right)$. We now use Veronese maps [17] to map the incidences between points and curves in $D$ to incidences between points and half-spaces in $\mathbb{R}^d$. More precisely, consider the map:

$$
\pi : p = (p_x, p_y) \in \mathbb{R}^2 \mapsto (x, x^2, \ldots, x^{2\delta}, y, yx, \ldots, yx^{\delta}, y^2) \in \mathbb{R}^d.
$$

We claim that for any curve $D \in D$, say defined by the equation $y = \sum_{i=0}^{\delta} a_i \cdot x^i$, there exists a half-space $H_D$ in $\mathbb{R}^d$ such that the set of points of $P$ contained in $D$ is precisely the set of points of $\pi(P)$ contained in $H_D$. The required half-space can be constructed as follows:

$$
p \in D \quad \text{if and only if} \quad \left( y - \sum_{i=0}^{\delta} a_i \cdot x^i \right) = 0$$

$$
\left( y - \sum_{i=0}^{\delta} a_i \cdot x^i \right)^2 \leq 0$$

$$
\left( a_0'x + a_1'x^2 + \cdots + a_{\delta}'x^{2\delta} \right) + \left( -2y \cdot (a_0x^0 + \cdots + a_\delta x^\delta) \right) + y^2 \leq a_0'
$$

for constants $a_0', \ldots, a_{\delta}'$ depending on $a_0, \ldots, a_\delta$. Labeling the coordinates in $\mathbb{R}^{3\delta+2}$ with $x_1, \ldots, x_{3\delta+2}$, the required half-space $H_D$ is then

$$
H_D : \quad a_0' \cdot x_1 + \cdots + a_{\delta}' \cdot x_{2\delta} + (-2a_0) \cdot x_{2\delta+1} + \cdots + (-2a_\delta)x_{3\delta+1} + x_{3\delta+2} \leq a_0',
$$

containing precisely the points that lie on the curve $D \in D$. This now implies a lower-bound of $\Omega\left(\frac{1}{\epsilon \left\lfloor \frac{d-2}{3} \right\rfloor}\right)$ for the $\epsilon$-Mnet for the primal set system induced by half-spaces in $\mathbb{R}^d$. Finally, the lower-bound follows for any value of $d$ by applying the bound for the largest $d' \leq d$ with integer value of $\frac{d-2}{3}$.

(b). By Veronese maps, points $P$ and disks $D$ can be lifted to half-spaces $H$ in $\mathbb{R}^3$ such that each point is lifted to a point in $\mathbb{R}^3$ and each disk is lifted to a half-space in $\mathbb{R}^3$ in such a way that their incidences are preserved. Now the required upper-bound follows from applying the bound in part (a) for half-spaces in $\mathbb{R}^3$ to the lifted point set of $P$.

(c). As a $k$-sided polygon can be partitioned into $k$ triangles, one of which must contain at least $\frac{\epsilon n}{k}$ points,
Consider any triangle \( T \) in the plane that contains \( en \) points of \( P \). By moving the sides of the triangle we can ensure that each side of \( T \) contains at least two points of \( P \) and this can be done in such a way that no point outside \( T \) enters the interior of \( P \). Some points in the interior of \( T \) may have moved to its boundary and some point outside \( T \) may also have moved to the boundary. Since at most 6 points may be on the boundary of \( T \), due to \( P \) being in general position, the interior of \( T \) still contains at least \( \frac{en}{2} \) points, assuming \( en \geq 12 \) (observe that for \( en < 12 \), the collection of singletons of \( P \) is an \( \epsilon \)-Mnet of size \( O(\frac{1}{\epsilon}) \)). Thus we can further restrict ourselves to the interior of triangles each of whose sides contain at least two points. The figure above shows a triangle with each side containing two points of \( P \). The points \( q \) and \( r \) could be identical, they could both be equal at the corner \( b \) of the triangle. Similarly \( s \) and \( t \) could be at \( c \) and \( u \) and \( p \) could be at \( a \). Observe that the triangles \( aqt, bsp, cur \) and \( prt \) cover the triangle \( T \) and therefore one of them must contain at least \( \frac{2n}{9} \) points of \( P \). Each of these triangles are of the following type: at least two of the corners are in \( P \) and all sides contain at least two points of \( P \). We call such triangles \emph{anchored} triangles. Thus we can again restrict ourselves to the problem of anchored triangles in the plane containing \( en \) points of \( P \).

Let \( O \) be the set of all anchored triangles for \( P \). Let \( O' = \{ \Delta_1, \ldots, \Delta_t \} \) be a maximal set of \( t \) triangles from \( O \) such that \( |\Delta_i \cap P| = en \) and \( |\Delta_i \cap \Delta_j \cap P| \leq \frac{en}{2} \).

**Lemma 4.1.** \( |O'| \leq 2 \cdot f_o \left( \frac{2}{n} \log \frac{1}{\epsilon}, 2c \log \frac{1}{\epsilon} \right) \), where \( f_o(m, l) \) is the maximum number of subsets of size at most \( l \) in the primal set system induced by objects in \( O \) on any subset of \( m \) points of \( P \), and \( c \) is some fixed constant.

**Proof.** Pick each point of \( P \) independently at random with probability \( p = \frac{c}{2en} \cdot \log \frac{1}{\epsilon} \) to get a random sample \( S \).

First, observe that with probability greater than \( \frac{1}{2} \), the sets \( \Delta_i \cap S, i = 1 \ldots t \), are distinct and \( |S| \leq \frac{2}{n} \log \frac{1}{\epsilon} \): consider the range space \( (P, R') \), where \( R' = \{ (\Delta_i \setminus \Delta_j) \cap P \mid \forall 1 \leq i < j \leq t \} \). From the definition of \( O' \), each set in \( R' \) has size at least \( en - \frac{en}{2} = \Theta(en) \). We now use the fact that ranges induced by polygons with \( k \) sides have VC dimension at most \( 2k + 1 \) \[17\]; it is easy to see that \( R' \) is a subset of the ranges induced by polygons (or union of polygons) with at most 9 sides, and so the VC dimension of \( R' \) is at most 19. Then by the Haussler-Welzl theorem \[19\], for \( c > 19 \cdot 4 \), with probability greater than \( \frac{3}{4} \), \( S \) is an \( \epsilon \)-net for \( (P, R') \). Now observe that if \( \Delta_i \cap S = \Delta_j \cap S \), then the set \( (\Delta_i \setminus \Delta_j) \cap S \) is empty, a contradiction to the fact that \( S \) is an \( \epsilon \)-net for \( R' \). From standard concentration estimates from Chernoff bounds, it follows that \( |S| \geq \frac{2}{n} \log \frac{1}{\epsilon} \) with probability less than \( \frac{1}{2} \).

For each \( \Delta_i \in O' \), let \( X_i \) be the random variable which is 1 if \( |\Delta_i \cap S| \geq 2c \log \frac{1}{\epsilon} \), and 0 otherwise. For a fixed \( i \), by linearity of expectation, we have \( E[|\Delta_i \cap S|] = \frac{2}{n} \log \frac{1}{\epsilon} \). By Markov’s inequality applied to each \( X_i \),

\[
Pr[X_i = 1] = Pr[|\Delta_i \cap S| \geq 2c \cdot \log \frac{1}{\epsilon}] = Pr \left[ |\Delta_i \cap S| \geq 4 \cdot E[|\Delta_i \cap S|] \right] \leq \frac{1}{4}.
\]

Hence \( E[Y] = E[\sum X_i] \leq \frac{t}{2} \), and by Markov’s inequality applied to \( Y \), we get that \( Pr \left[ \sum X_i \geq \frac{t}{2} \right] \leq \frac{1}{2} \).

We can conclude that there exists a subset \( S \) of size \( \frac{2}{n} \log \frac{1}{\epsilon} \) such that \( \Delta_i \cap S \) are distinct for all objects in \( O' \), and for at least \( \frac{|O'|}{2} \) of the objects in \( O' \), we have \( |\Delta_i \cap S| \leq 2c \log \frac{1}{\epsilon} \). Therefore we can get the required bound on the size of \( O' \):

\[
\frac{|O'|}{2} \leq f\left(|S|, l\right) = f\left(\frac{c}{n} \log \frac{1}{\epsilon}, 2c \log \frac{1}{\epsilon}\right).
\]

**Remark:** After the appearance of the conference version of this paper, the statement of Lemma 4.1 has been formalized as the shallow packing lemma. We refer the reader to \[17\] for details and recent history.
We will need the following theorem from [14].

**Theorem C** (Simplicial partition theorem). *Given a set \( P \) of \( n \) points in \( \mathbb{R}^d \), and an integer parameter \( t > 0 \), there exists a partition of \( P \) into \( t \) sets, each of size \( \Theta(\frac{n}{t}) \), such that any hyperplane intersects the convex-hull of at most \( O(t^{1 - 1/d}) \) sets of the partition.*

Take this set \( O' \) of maximal objects, each containing \( en \) points of \( P \), and every pair of objects in \( O' \) intersecting in less than \( \frac{en}{2} \) points. For each object \( \Delta_i \in O' \), do the following: apply the simplicial partition theorem to \( \Delta_i \cap P \) with the parameter \( t \), set to a large enough constant, to get a partition of \( \Delta_i \cap P \) into \( t \) sets of size \( \Theta(\frac{|\Delta_i \cap P|}{t}) \). Add each of these \( t = O(1) \) sets to the \( \epsilon \)-Mnet \( M \) for \( P \).

**Claim 5.** \( M \) is an \( \epsilon \)-Mnet for the primal set-system induced by \( O \), of size \( O\left(f_O\left(\frac{\epsilon}{t} \cdot \log \frac{1}{\epsilon}, 2c \log \frac{1}{\epsilon}\right)\right) \).

**Proof.** First note that each set added to \( M \) had size \( \Theta(\frac{|\Delta_i \cap P|}{t}) = \Theta(en) \), and the number of such sets is \( O(|O'| \cdot t) = O(|O'|) \). It remains to show that any object containing \( en \) points of \( P \) contains one set of \( M \). Take any triangle \( \Delta \) containing \( en \) points of \( P \) (any triangle containing greater than \( en \) points can always be shrunk to a triangle containing fewer points). By the maximality of \( O' \), there exists \( \Delta_i \in O' \) such that \( |\Delta \cap \Delta_i| \geq \frac{en}{2} \). Furthermore, of all the sets in the simplicial partition of \( \Delta_i \), each edge of \( \partial \Delta \) can intersect only \( O(\sqrt{t}) \) sets; so in total the three bounding segments of \( \Delta \) can intersect at most \( O(3 \sqrt{t}) \) sets. Each of these sets has \( O\left(\frac{|\Delta_i \cap P|}{t}\right) \) points. So these sets can contribute at most \( O(3 \sqrt{t} \cdot \frac{|\Delta_i \cap P|}{t}) \) points of \( \Delta_i \) to \( \Delta \). Setting \( t \) to be a large-enough constant (say, \( t = 38 \)), this is less than \( \frac{en}{2} \). Therefore \( \Delta \) must contain a point in \( \Delta_i \) which lies in a partition for \( \Delta_i \) not intersecting \( \partial \Delta \), i.e., the partition lies completely inside \( \Delta \). \( \square \)

Finally, when \( O \) is a set of anchored triangles in the plane, a routine application of the Clarkson-Shor method [17] implies that \( f_O(n, l) = O(n^3 \cdot l) \). Then Lemma 5 implies the existence of \( \epsilon \)-Mnets for the primal set system induced by \( O \) of size \( O\left((\frac{\epsilon}{t} \log \frac{1}{\epsilon})^3 \cdot 2c \log \frac{1}{\epsilon}\right) = O\left(\frac{1}{\epsilon^4} \left(\log \frac{1}{\epsilon}\right)^4\right) \).

**d.** The upper-bounds for the primal set systems induced by lines, strips, cones in the plane again follow from Lemma 5. The function \( f(n, l) \) correspondingly denotes the number of subsets of size \( l \) induced by the objects of the appropriate type (lines, strips, cones). For lines, \( f(n, l) = O(n^2) \) implies the existence of \( \epsilon \)-Mnets of size \( O\left(\frac{1}{\epsilon^2} \left(\log \frac{1}{\epsilon}\right)^2\right) \); for strips \( f(n, l) = O(n^2 \cdot l) \) implies the existence of \( \epsilon \)-Mnets of size \( O\left(\frac{1}{\epsilon^2} \left(\log \frac{1}{\epsilon}\right)^3\right) \); and for cones, \( f(n, l) = O(n^2 \cdot l^2) \) implies the existence of \( \epsilon \)-Mnets of size \( O\left(\frac{1}{\epsilon^2} \left(\log \frac{1}{\epsilon}\right)^4\right) \).

The lower-bound for the primal set system induced by lines, strips and cones in the plane follows from Theorem 4 by setting \( \delta = 1 \):

**Corollary 4.2.** For any \( \epsilon > 0 \) and integer \( n \), there exists a set \( P \) of \( n \) points in the plane such that any \( \epsilon \)-Mnet for the primal set system on \( P \) induced by lines must have size \( \Omega\left(\frac{1}{\epsilon^2}\right) \).

As the set system induced by lines is a special case for the ones induced by strips and cones, this implies the same lower-bound for the primal set system induced by strips and cones in the plane.

**e.** As each rectangle contains \( en \) points of \( P \) and intersects the \( y \)-axis, for each rectangle \( R \), take the portion of the rectangle on the side of the \( y \)-axis that contains at least \( \frac{en}{2} \) points. We can construct \( \frac{\epsilon}{t} \)-Mnets for the two sides of the \( y \)-axis separately and return the union of the two Mnets. Now for the primal set system induced by axis-parallel rectangles with one vertical edge lying on the \( y \)-axis, we have \( f(n, l) = O(n) \). Now Lemma 5 implies that one can construct \( \frac{\epsilon}{t} \)-Mnets of size \( O\left(\frac{1}{\epsilon}\right) \).
5 Conclusion and future work

We conclude our study by observing that the above series of results—with proofs that use different techniques—indicate an intriguing relation between the sizes of $\epsilon$-nets and the sizes of $\epsilon$-Mnets. In all cases, they obey the following pattern: if there exist $\epsilon$-nets of size $O\left(\frac{1}{\epsilon} f\left(\frac{1}{\epsilon}\right)\right)$ for some primal or dual set system, then the size of $\epsilon$-Mnets for the same set system is $O\left(\frac{1}{\epsilon} c f\left(\frac{1}{\epsilon}\right)\right)$, where $c$ is some constant. For example, for all spaces known to have linear-sized $\epsilon$-nets (which is optimal), our proofs establish the existence of linear-sized $\epsilon$-Mnets (which is optimal). For the primal set system induced by axis-parallel rectangles in the plane, $\epsilon$-nets have size $O\left(\frac{1}{\epsilon} \log \log \frac{1}{\epsilon}\right)$ (shown to be optimal) [21]; our results show the existence of $\epsilon$-Mnets of size $O\left(\frac{1}{\epsilon} \log \frac{1}{\epsilon}\right)$ (which we show to be optimal). For the primal set system induced by half-spaces in $\mathbb{R}^d$, $\epsilon$-nets have size $O\left(\frac{d}{\epsilon} \log \frac{1}{\epsilon}\right)$ (shown to be optimal [12]); our results establish the existence of $\epsilon$-Mnets for this set system of size $O\left(\frac{1}{\epsilon^{(d+1)/3}}\right)$. Similarly, for the remaining set systems for which there exist $\epsilon$-nets of size $O\left(\frac{1}{\epsilon} \log \frac{1}{\epsilon}\right)$, we show the existence of $\epsilon$-Mnets of size $O\left(\frac{1}{\epsilon}\right)$. It would be interesting to see if there is any connection with the (still) open problem of finding the right bound on the size of $\epsilon$-nets for the primal set system induced by lines in the plane.

References


