Majority Consensus and the Local Majority Rule

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Abstract. We study a rather generic communication/coordination/computation problem: in a finite network of agents, each initially having one of the two possible states, can the majority initial state be computed and agreed upon by means of local computation only? We describe the architecture of networks that are always capable of reaching the consensus on the majority initial state of its agents. In particular, we show that, for any truly local network of agents, there are instances in which the network is not capable of reaching such consensus. Thus, every truly local computation approach that requires reaching consensus is not failure-free.

1 Introduction

Attempting to solve a complex problem by a simultaneous coordinated activity of local agents is an idea that arises naturally in a variety of contexts. For example, this idea is fundamental in frameworks as diverse as distributed computing and neural networks. While methods of local computation and decision-making are often effective in dealing with complex tasks, the successful implementation of such methods often raises a new breed of problems related to coordination and communication of local agents.

We study a discrete time, memoryless, synchronous dynamic process and call it local majority process on (a finite network) $G$. Informally, the vertices of a graph $G = (V, E)$ represent the agents and the edges of $G$ represent all (bidirectional) communication links between pairs of agents. Initially, at time $t = 0$, each agent is in one of the two possible states, e.g., colored red or blue (voted Yes or No, having value 0 or 1, . . .). Then the local majority rule is applied synchronously and iteratively: an agent has different colors at time $t$ and $t + 1$ if and only if the agent’s color at time $t$ is not a majority color in the agent’s neighborhood in $G$ at time $t$. A precise formulation of the model will be given in the next section.

The local majority process (and some of its natural extensions) has been studied in frameworks as diverse as social influence [PS83, PT86a, PT86b] and neural networks [GO81, GO80, Gol86, GM90]. Recently, the local majority process has
reappeared (under the name polling process) in several papers motivated by certain distributed computing problems [Pel98, Ber99, FLL98, FLL+99, Has98, HP99, HP00, LPS99, NIY99, NIY00].

A natural question to ask is when does the local majority process ensure that all agents reach a consensus on the initial majority state? We will say that $G$ is a majority consensus computer (m.c.c.) if, for any set of initial states (there are $2^{|V|}$ such sets), the local majority process simultaneously brings all agents into the state that was the initial majority state. Note that, according to the local majority process, once all agents are in the same state, no agent will change its state ever after. All of the recent papers dealing with the local majority process and its modifications [Pel98, Ber99, FLL98, FLL+99, Has98, HP99, HP00, LPS99, NIY99, NIY00] investigated how badly could the local majority process (and its variations) miscalculate the initial majority (on a specific class of graphs)\footnote{For example, Berger [Ber99] has shown that for every $n$ there exists a $G$ on at least $n$ vertices and the set of states such that only 18 vertices are in one state and the rest are in the other, yet the local majority process forces all vertices to simultaneously end up in the initial minority state.} In contrast to these results, we are interested in $G$ which are immune to miscalculations in the local majority process, i.e., the focus of this paper are majority consensus computers and investigation of their structure.

Since being a majority consensus computer is seemingly a very strong property, one would expect that a sort of an impossibility theorem holds. As will be shown, the situation is not that simple and the full characterization of m.c.c.'s remains an open problem. However, our results demonstrate in several ways that the non-locality is inherent property of every m.c.c. Thus, reaching consensus on the majority is a truly non-local task in the sense that a most natural local computation procedure is failure-free only if computing local majority is essentially as complex as computing global majority.

2 Majority Consensus Computers

A standard graph theoretic notation is used throughout the paper. $G = (V, E)$ denotes an undirected, simple, finite graph $G$ with the vertex set $V$, $|V| = n$, and the edge set $E$ (i.e. $E \subseteq \{S \subseteq V : |S| = 2\}$). The neighborhood of vertex $v$ in set $S \subseteq V$ is the set of all neighbors of $v$ that are in $S$, $N_S(v) := \{s \in S : \{v, s\} \in E\}$, and the degree of $v$ in $S$ is $deg_S(v) = |N_S(v)|$ (if $S = V$, the subscript is omitted). Min and max degree in $G$ are denoted by $\Delta(G) = \max\{deg(v) : v \in V\}$ and $\delta(G) = \min\{deg(v) : v \in V\}$. Some non-standard terminology: a vertex $v$ is a master if $deg(v) = n-1$ (i.e., $v$ is adjacent to every other vertex). We also say that $v$ is a $k$-master if $deg(v) = n-1-k$ (i.e., $v$ is adjacent to all but $k$ other vertices). Note that 0-master and master are equivalent notions, and we will use them interchangeably throughout the rest of the text.

In our model all agents and communication links in the system are represented by a graph $G$ in a natural way. That is, the vertices of $G$ are in a
one-to-one correspondence with the agents and the edges of $G$ correspond to adjacency relations among the agents.

A coloring of the graph $G$, $c^t : V \to \{0, 1\}$ defines an assignment of binary values (colors) to the vertices of $G$ at time $t$. We use the notation $c^t_v := c^t(v)$ to denote the color of a vertex $v$ at time $t$. The notation $\text{sum}(c^t) := \sum_{v \in V} c^t_v$ will also be useful. A color which is assigned to more than $|V|/2$ vertices at a time $t$ is called the majority color of the coloring $c^t$ and denoted by $\text{maj}(c^t)$. Thus, $\text{maj}(c^t) = 1$ if and only if $\text{sum}(c^t) > n/2$ and $\text{maj}(c^t) = 0$ if and only if $\text{sum}(c^t) < n/2$. Note that $\text{maj}(c^t)$ is not defined if $|V|$ is even and $c^t$ defines an equipartition of $V$, i.e., if $\text{sum}(c^t) = n/2$. A coloring $c^t$ is a consensus if it is constant, i.e. if all the vertices of $G$ have the same binary values (colors). Thus, $c^t$ is a consensus if and only if $c^t_v = \text{maj}(c^t)$ for all $v \in V$. We will sometimes abuse the notation and write $c^t = 0$ or $c^t = 1$ for consensus in color 0 and 1, respectively. Another abuse of notation is $(1-c^t)$ denoting the coloring obtained from $c^t$ by changing the color of every vertex, i.e., for every $v \in V$ and coloring $c^t$, $(1-c^t)_v = 1 - c^t_v$.

The main object of our study is the local majority process $LMP(G, c^0)$, a discrete time process on $G$ that is based on the iterative application of the local majority rule. The process is completely defined by $G$ and the initial coloring $c^0$. For every $t = 0, 1, 2, \ldots$, the coloring $c^{t+1}$ is derived by applying the local majority rule on $N(v)$ for each vertex in $G$:

$$c^{t+1}_v = \begin{cases} c^t_v & \text{if } |\{w \in N(v) : c^t_w = c^t_v\}| \geq |N(v)|/2 \\ 1 - c^t_v & \text{if } |\{w \in N(v) : c^t_w \neq c^t_v\}| > |N(v)|/2 \end{cases}$$ (1)

The local majority rule simply states that, at the next discrete time step, the color assigned to a vertex $v$ will be the color of the majority of its neighbors. Note that an even degree vertex will retain its color whenever exactly half (or more) of its neighbors have the same color. The above rule also implies that the local majority rule is executed simultaneously for all the vertices. We say that there is a majority switch at time $(t + 1)$ if $\text{maj}(c^t) \neq \text{maj}(c^{t+1})$.

Note that if $c^t$ is a consensus, then $c^{t+k} = c^t$ for all positive integers $k$.

If, for some positive integer $t$, $c^t$ is a consensus, then we say that $G$ reaches consensus for $c^0$. If $G$ reaches consensus $c^t$ for $c^0$ and $c^t = \text{maj}(c^0)$, then we say that the $LMP(G, c^0)$ correctly computes the initial majority and that $G$ admits a majority consensus for the initial coloring $c^0$. A graph $G$ is a majority consensus computer (or a m.c.c. in short) if, for every $c^0$ (there are $2^n$ such colorings), $LMP(G, c^0)$ correctly computes the initial majority. In other words, $G$ is a m.c.c. if $G$ admits majority consensus for all of the $2^n$ possible initial colorings. Note that for every graph with even number of vertices there exists a $c^0$ where $\text{maj}(c^0)$ is not defined. Thus, $G$ can be a majority consensus computer only if it has an odd number of vertices. **Throughout the rest of the paper we assume that $n$ is odd.**

Our first observation about majority consensus computers is the following proposition.\(^2\)

\(^2\) The proofs omitted throughout the paper can be found in [MP00].
**Proposition 1.** Let \( G \) be a m.c.c. and let \( c^0 \) be an initial coloring of \( G \). Then there are no majority switches for \( LMP(G,c^0) \), i.e., \( \text{maj}(c^t) = \text{maj}(c^0) \) for \( t = 0,1,2,\ldots \).

Next note that there are only \( 2^n \) possible colorings and \( c^{t+1} \) is a function of \( G \) and \( c^t \), thus the sequence \( c^0,c^1,c^2,\ldots \) must become periodic, i.e., there exists positive integers \( t_0 \) and \( k \) such that \( c^{t+k} = c^t \) for every \( t \geq t_0 \). Obviously, the period \( k \) and \( t_0 \) are not larger than \( 2^n \). Somewhat surprisingly, the period can be only one or two and there exists \( t_0 \) smaller than \( |E| \).

**Theorem 1.** Consider the sequence \( c^0,c^1,c^2,\ldots \) defined by the local majority process on \( G \) with initial coloring \( c^0 \), \( LMP(G,c^0) \). Then there exists \( t_0 < |E| \) such that \( c^t = c^{t+2} \) for every \( t \geq t_0 \).

Many of our results will be based on the “period is at most two” property. Next we show that a monotonicity property with respect to the structure of the coloring holds in the local majority process.

**Lemma 1.** Let \( V_i(c^t) = \{ v \in V : c^t_v = i \} \), \( i = 0,1 \), where \( c^t \) is a coloring of \( G = (V,E) \). If there exists \( i \in \{0,1\} \) and colorings \( c^t \) and \( d^t \) such that \( V_i(c^t) \subseteq V_i(d^t) \) then \( V_i(c^{t+k}) \subseteq V_i(d^{t+k}) \) for \( k = 0,1,2,\ldots \).

According to the definition, in order to check whether \( G \) is a majority consensus computer, one would have to check whether \( G \) admits majority consensus for all \( 2^n \) possible initial colorings \( c^0 \). However, because of the monotonicity property described in Lemma 1, it suffices to consider only colorings \( c^0 \) such that \( \text{sum}(c^0) = (n+1)/2 \). (There are \( \binom{n}{(n+1)/2} = O(2^n/\sqrt{n}) \) such colorings).

**Theorem 2.** \( G \) is a majority consensus computer if and only if \( G \) admits majority consensus for any \( c^0 \) such that \( \text{sum}(c^0) = (n+1)/2 \).

**Remark.** Unfortunately, it is not true that adding an edge to or deleting an edge from a majority consensus computer \( G \) preserves the property “majority consensus computer”. For example, consider \( (K_n)^c \subseteq K_n \setminus P_{n-1} \subseteq K_n \setminus P_{(n+1)/2} \subseteq K_n \) where \( n \) is odd. It is not difficult to show that \( K_n \) and \( K_n \setminus P_{n-1} \) are m.c.c., while \( (K_n)^c \) and \( K_n \setminus P_{(n+1)/2} \) are not (see [MP00]).

We close this section by showing that masters in \( G \) compute majority instantly, i.e., the color of a master at time \( t+1 \) is \( \text{maj}(c^t) \). (Larger the difference between the majority and minority color of \( c^t \), smaller degree of \( v \) is needed to ensure \( c^{t+1}_v = \text{maj}(c^t) \).)

**Proposition 2.** If \( v \) is a master in \( G \), then \( c^{t+1}_v = \text{maj}(c^t) \). More generally, if \( v \) is a \( k \)-master in \( G \) and \( |\text{sum}(c^t) - n/2| \geq (k + 1)/2 \), then \( c^{t+1}_v = \text{maj}(c^t) \).

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\(^3\) This theorem is a straightforward consequence of a much more general result that can be found in, e.g., [GM90] (various variations and extensions can be found in this rather comprehensive collection of results related to dynamic behavior of neural and automata networks); sufficient conditions for the property in the case of LMP on infinite graphs were studied in [Mor94b, Mor94a, Mor95].
3 Structural properties

Let's start by presenting a class of graphs that are m.c.c. and a class of graphs that are not m.c.c..

**Proposition 3.**
(a) A graph $G$ with more than $n/2$ masters is a majority consensus computer.
(b) A graph $G$ with exactly $(n-1)/2$ masters is not a majority consensus computer.

Next we give a characterization of majority consensus computers which indicates a way towards a static representation in the form of existence of a particular partition of the vertices of $G$.

**Theorem 3.** $G$ is not a majority consensus computer if and only if at least one of the following holds:
(a) There exists $c^0$ such that $\text{maj}(c^0) \neq \text{maj}(c^1)$
(b) There exists a partition of $V$ into four sets $A_0$, $A_1$, $B_0$, $B_1$ satisfying

1. $|B_0||B_1| = 0 \Rightarrow |A_0||A_1| \geq 1$,
2. For every $v \in A_i$, $i = 0, 1$: $\deg_{A_i}(v) - \deg_{A_{3-i}}(v) \geq |\deg_{B_i}(v) - \deg_{B_{3-i}}(v)|$
3. For every $v \in B_i$, $i = 0, 1$: $\deg_{B_i}(v) - \deg_{B_{3-i}}(v) > |\deg_{A_i}(v) - \deg_{A_{3-i}}(v)|$

**Proof.** Suppose $G$ is not a majority consensus computer. If $G$ admits a consensus for every possible initial coloring $c^0$, there must exist $d^0$ for which $G$ does not admit a majority consensus, i.e., there exists $d^0$ and $t$ such that $d^t$ is a consensus and $\text{maj}(d^0) \neq \text{maj}(d^t)$. Obviously, in the sequence $d^0, d^1, \ldots, d^t$, there exists $t' < t$ such that $\text{maj}(d^{t'}) \neq \text{maj}(d^{t'+1})$. Thus, (a) holds for $c^0 := d^{t'}$.

Thus, we may assume that there exists $d^0$ for which $G$ does not admit a consensus. By Theorem 1 there exists $t$ such that $c^t = c^{t+2}$. For $i = 0, 1$ define $A_i := \{v \in V : i = c^t = c^{t+1}\}$ and $B_i := \{v \in V : i = c^t \neq c^{t+1}\}$. Note that $A_0$, $A_1$, $B_0$, $B_1$ partition $V$ and that $1$ must hold since neither $c^t$ nor $c^{t+1}$ is a consensus. Since for every $v \in A_i$, $c^t = c^{t+1}$, $\deg_{A_i}(v) + \deg_{B_i}(v) \geq \deg_{A_{3-i}}(v) + \deg_{B_{3-i}}(v)$. Similarly, for every $v \in A_i$, $c^{t+1} = c^{t+2}$ implies (because $\{v : c^t = i\} = A_i \cup B_{3-i}$) $\deg_{A_i}(v) + \deg_{B_i}(v) \geq \deg_{A_{3-i}}(v) + \deg_{B_{3-i}}(v)$. These two inequalities imply 2. In the same manner, it follows that for every $v \in B_i$, $c^t \neq c^{t+1}$ implies $\deg_{A_{3-i}}(v) + \deg_{B_{3-i}}(v) > \deg_{A_i}(v) + \deg_{B_i}(v)$ and that $c^{t+1} \neq c^{t+2}$ implies $\deg_{A_i}(v) + \deg_{B_i}(v) > \deg_{A_{3-i}}(v) + \deg_{B_{3-i}}(v)$. Hence, 3. follows from these two inequalities.

The converse is straightforward to verify. (If (b) holds, set $c^0 = i$ for $v \in A_i \cup B_i$.)

The last theorem indicates that majority consensus computers are highly connected graphs. For example, it follows in a straightforward manner that if a graph is bipartite or disconnected, then it is not a majority consensus computer. Furthermore, it can be shown that any majority consensus computer has trivial min-cuts, and no unique max-cuts. The following theorem and its corollary provide another confirmation of this claim.
Theorem 4. Let $G$ be a majority consensus computer. Then for every $v \in V$

$$| \bigcup_{w \in N(v)} N(w)| \geq n/2. \quad (2)$$

Proof. First note that we can assume that $G$ is connected and that $n > 2$.

Suppose (2) does not hold for some $v \in V$. Let $u \in V$ be a vertex of the minimum degree among all vertices $v$ for which (2) is violated. Let $c^0$ be such that $c^0_v = 1$ for every $v \in \bigcup_{w \in N(u)} N(w)$ and such that $\text{sum}(c^0) = (n + 1)/2$. Note that $c^0_v = 1$ and that $\text{maj}(c^0) = 1$. Let $d^0$ be such that $d^0_v \neq c^0_v$ if and only if $v = u$ (i.e., the only difference between $c^0$ and $d^0$ is in the color of $u$). Note that $\text{sum}(d^0) = (n - 1)/2$ and thus

$$\text{maj}(d^0) = 0 \neq 1 = \text{maj}(c^0). \quad (3)$$

Observe that for all $v \notin N(v) \cup \{u\}, w \in N(v) \Rightarrow c^0_w = d^0, and hence \ c^1 = d^0$. Further observe that $c^1_v = d^1_v = 1$ because the color of all neighbors of $u$ is 1 in both $c^0$ and $d^0$ (and $u$ has at least one neighbor since $G$ is connected). Finally observe that by the choice of $u$ and the fact that $G$ is connected and $n > 2$, $\deg(v) \geq 2$ for all $v \in N(u)$. Since the color of all neighbors of $v$ other than $u$ is 1 in both $c^0$ and $d^0$, it follows that $c^1_v = d^1_v$ for $v \in N(u)$. Hence, $c^1 = d^1$ and thus, because of (3), either $\text{maj}(c^1) \neq \text{maj}(c^0)$ or $\text{maj}(d^0) \neq \text{maj}(d^0)$. In either case, it follows from Proposition 1 that $G$ is not a majority consensus computer.

The theorem shows that majority consensus computers are nowhere truly local since the second neighborhood of any vertex contains a majority of the vertices of $V$. Hence, the local majority process always reaches a consensus on the initial majority color only if the local majority rule is nowhere local. Hence, the theorem can be viewed as a sort of an impossibility result.

Corollary 1. If $G$ is a majority consensus computer then $\text{diam}(G) \leq 4$, i.e., the length of the shortest path between any two vertices is at most 4.

Exhaustive computer aided search confirmed that $\text{diam}(G) \leq 2$ for every majority consensus computer on at most 13 vertices. We conjecture that a much stronger statement is true (also confirmed to hold for $n \leq 13$ by an exhaustive search method).

**Master Conjecture.** Every majority consensus computer contains a master.

This is a rather strong conjecture because it implies that a necessary condition for reaching majority consensus is the existence of a vertex connected to all the other vertices, thereby annihilating any notion of local computation.

The master conjecture also has interesting implications. For example, it can be shown that any majority consensus computer with $(n - 1)/2$ master vertices has minimum degree at least $(n - 1)$. Thus the minimum degree of every vertex is strongly related to the number of master vertices in the graph.
4 Master conjecture for highly connected graphs

In this section we show that the master conjecture holds for graphs $G$ with $\delta(G) \geq n - 3$. Note that, intuitively, such graphs should be considered as prime candidates for a counterexample to the conjecture since all of the vertices in these graphs are either masters or very close to being masters (i.e., 0-masters, 1-masters, or 2-masters).

A direct consequence of Proposition 2 is that the only colorings $c^0$ for which $G$ might not admit a majority consensus are the tight ones, i.e., $c^0$ such that $\text{sum}(c^0) = (n + 1)/2$. (The case $\text{sum}(c^0) = (n - 1)/2$ is symmetric).

Proposition 4. If $\delta(G) \geq n - 3$, then $G$ admits majority consensus for every $c^1$ such that $\text{sum}(c^0) \geq (n + 3)/2$.

If $\delta(G) \geq n - 3$, then $G^c$ has a very simple structure since $\Delta(G^c) = (n - 1) - \delta(G) \leq (n - 1) - (n - 3) = 2$. In other words, a connected component of $G^c$ is a single vertex, a path, or a cycle. The decomposition of $G^c$ into its connected components $H_1 = (V_1, E_1^c), H_2 = (V_2, E_2^c), \ldots, H_m = (V_m, E_m^c), \ldots$ will be used throughout this section and we will often abuse the notation and identify $V(H)$ with $H$ whenever such notation will be unambiguous (e.g., we will often say that the connected components of $G^c$ define a partition of $V$).

Another convenient property of $G$ with $\delta(G) \geq n - 3$ is that every vertex in $G$ is either a master, a 1-master, or a 2-master. Thus, the following lemma gives a complete boolean formula representation of local changes for colorings $c^1$ with $\text{sum}(c^0) = (n + 1)/2$.

Lemma 2. Let $c^1$ be such that $\text{sum}(c^0) = (n + 1)/2$.

(a) If $v$ is a master, then $c^1_{v+1} = 1$.

(b) If $v$ is a 1-master, then $c^1_{v+1} = 1 - c^1_v c^1_w$ where $w$ is the unique vertex not adjacent to $v$.

(c) If $v$ is a 2-master, then $c^1_{v+1} = 1 - c^1_u c^1_w$ where $u$ and $w$ are the two vertices not adjacent to $v$.

Proof. If $v$ is a 2-master then $V \setminus N(v) = \{v, u, w\}$, so

$$|\{u \in N(v) : c^1_u = 1\}| = \frac{n + 1}{2} - c^1_v - c^1_u - c^1_w$$

First suppose $c^1_v = 0$. Then, $c^1_{v+1} = 0$ if and only if $|\{u \in N(v) : c^1_u = 1\}| < |N(v)|/2 = (n - 3)/2$ and this is true if and only if $c^1_u = c^1_w = 1$. Thus, (c) holds if $c^1_v = 0$. Finally, suppose $c^1_v = 1$. Then, $c^1_{v+1} = 0$ if and only if $|\{u \in N(v) : c^1_u = 1\}| < |N(v)|/2 = (n - 3)/2$ and, again, this is true if and only if $c^1_u = c^1_w = 1$. Thus, (c) also holds if $c^1_v = 1$. The proof of (b) is similar, while (a) is obvious. □

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4 In other words, $V_i, i = 1, \ldots, m$ are pairwise disjoint, $V_1 \cup \ldots \cup V_m = V$, and $E_i^c \cup \ldots \cup E_m^c = E^c$. 

This lemma allows us to track the action of the local majority process on $G$. We define an auxiliary graph $AG = (V, E(AG))$. The edges of $AG$ are defined using the formulas from (b) and (c) from the lemma: $E(AG) = \{ (v, w) : d_C(v) = n-2, (v, w) \notin E(G) \} \cup \{ (v, w) : d_C(v) = n-3, (v, w) \notin E(G) \}$. Thus, $E(AG)$ is in one to one correspondence with the set of all vertices of $G$ which are not masters. Note that $AG$ has a rather simple structure: all of its connected components are cycles, each corresponding to a connected component of $G^c$ as follows (this is a direct consequence of the definition of $AG$):

If a connected component $H \subseteq G^c$ is a path, say $v_1, v_2, \ldots, v_l$ (i.e., $\{v_i, v_{i+1}\} \in E(G^c), i = 1, \ldots, l-1$), then $V(H)$ defines a cycle $C_H$ that is a connected component in $AG$: If $l$ is even, the adjacent vertices in $C_H$ are $v_1, v_2, v_4, \ldots, v_{l-2}, v_{l-1}, v_l$. If $l$ is odd, the adjacent vertices in $C_H$ are $v_1, v_2, v_4, \ldots, v_{l-1}, v_l, v_2, v_4, \ldots, v_{l-3}, v_3, v_1$.

If a connected component $H \subseteq G^c$ is an odd cycle (i.e. odd number of vertices), say $v_1, v_2, \ldots, v_{2k+1}, v_1$ (i.e., $\{v_i, v_{i+1}\} \in E(G^c), i = 1, \ldots, 2k+1$, and $\{v_{2k+1}, v_1\} \in E(G^c)$), then $V(H)$ defines a cycle $C_H$ that is a connected component in $AG$: $v_1, v_3, \ldots, v_{2k+1}$. $v_1$ and $v_2, v_4, \ldots, v_{2k}, v_2$.

**Lemma 3.** Let $c^l$ such that $\text{sum}(c^l) = (n + 1)/2$. Let $H$ be a connected component of $G^c$ on $l$ vertices, $l \geq 2$. Let $S = \{v \in H : c^l_v = 1\}$. Then

$$|\{v \in H : c^l_v = 1\}| \geq l - |S|$$

(4)

Furthermore, the equality holds in (4) if and only if one of the following holds: (i) $|S| = 0$, (ii) $|S| = l$, (iii) $H$ is an even cycle and $c^l_v \neq c^l_w$ whenever $\{v, w\} \in E(G^c)$.

**Proof.** First note that, by Lemma 2 and by definition of $AG$,

$$|\{v \in H : c^l_v = 1\}| = \sum_{v \in H} c^l_v = \sum_{\{u, w\} \in C \cap AG} (1 - c^l_w) = |H| - \sum_{\{u, w\} \in C \cap AG} c^l_u c^l_w.$$  

Thus, it remains to show that

$$|S| \geq \sum_{\{u, w\} \in C \cap AG} c^l_u c^l_w.$$

(5)

Note that

$$\sum_{\{u, w\} \in C \cap AG} c^l_u c^l_w = |\{\{u, w\} \in E(AG) : u, w \in S\}| = |E(C_H[S])|$$

where $C_H[S]$ denotes the induced subgraph of $C_H$, i.e., the maximal subgraph of $C_H$ on the vertex set $S \subseteq V(C_H)$. If $|S| = 0$, $|E(C_H[S])| = 0$, and (5) holds with equality. Thus, (4) holds with equality. If $|S| = l$, then $C_H[S] = C_H$.
and $|E(C_H[S])| = |E(C_H)| = l$ since $C_H$ is a cycle or a union of two disjoint cycles. Thus, if $|S| = l$, (4) also holds with equality. If $H$ is an even cycle, then $C_H = C_1 H \cup C_2 H$. Furthermore, $S = V(C_1 H)$ or $S = V(C_2 H)$ if and only if vertices of $H$ are colored alternately along the cycle $H$ (i.e., as described in (iii) in the statement of the lemma). In either case, $|E(C_H[S])| = |S|$ and (4) again holds with equality. If neither (i) nor (ii) nor (iii) holds, then $C_H[S]$ contains an acyclic component and any possible cyclic component of $C_H$ must be a cycle.\footnote{In fact the only possibility for a cyclic component is when $H$ is an even cycle.}

Thus, $|E(C_H[S])| \leq |S| - 1$ and $|\{v \in H : c_v^{i+1} = 1\}| \geq l - |S| + 1$. □

Several simple consequences of this lemma will be useful in the analysis that follows. For example, if a connected component of $G^c$ that is not an isolated vertex is monochromatic for some $c^i$, then every vertex in $H$ will switch color.

**Lemma 4.** Let $c^i$ be a coloring of $G$, $\delta(G) \geq n - 3$, such that $\text{sum}(c^i) = (n + 1)/2$. Let $H = (V_H, E_H)$ be a connected component of $G^c$ with $|V_H| \geq 2$. Suppose that $c_v^i = c_w^i$ for every $v, w \in V_H$. Then $c_v^{i+1} = 1 - c_v^i$ for every $v \in V_H$.

The next lemma presents an opposite scenario: if colors assigned by $c^i$, $\text{sum}(c^i) = (n + 1)/2$, alternate along an even cycle that is a connected component of $G^c$, then no vertex on that cycle will switch color.

**Lemma 5.** Let $c^i$ be a coloring of $G$, $\delta(G) \geq n - 3$, such that $\text{sum}(c^i) = (n + 1)/2$. Let $C_{2k} \subset G^c$ be a connected component in $G^c$. Suppose the colors assigned by $c^i$ alternate along the cycle: if $u$ is adjacent to $v$ in $C_{2k}$ then $c_u^i = 1 - c_v^i$. Then $c_v^{i+1} = c_v^i$ for every $v \in C_{2k}$.

The preceding lemmas indicate a way to construct $c^0$ yielding a complete switch, i.e., $c^1 = 1 - c^0$. Obviously, all masters must be colored with a minority color in order to switch. If all the other connected components of $G^c$ are monochromatic (with some even cycles possibly being colored as described in the previous lemma), and if the resulting coloring $c^0$ is a tight majority coloring on $G$ (i.e., $\text{sum}(c^0) = (n + 1)/2$), then, as shown in the next lemma, $c^1 = 1 - c^0$ (except on the even cycles where $c^{d+1} = c^d$), and $G$ is not a majority consensus computer.

**Lemma 6.** Let $\delta(G) \geq n - 3$. Let $H_1 = (V_1, E_1^1)$, $H_2 = (V_2, E_2^2)$, ..., $H_m = (V_m, E_m^m)$ be the connected components of $G^c$. Suppose there exist $i$ and $j$, $1 \leq i < j \leq m$, such that

(i) $|V_k| = 1 \Rightarrow k \leq i$, 
(ii) $m \geq k > j \Rightarrow H_k$ is an even cycle, 
(iii) $|V_1| + |V_2| + \ldots + |V_i| + 1 = |V_{i+1}| + \ldots + |V_j|$. 

Then $G$ is not a majority consensus computer.

**Proof.** For $v \in V_k$, set $c_v^0 = 0$ if $k \leq i$ and set $c_v^0 = 1$ if $i < k \leq j$. If $j < m$, then the remaining vertices lie on even cycles in $G^c$. Color each $H_k$ alternately, i.e., as described in the statement of Lemma 5. Note that, by (iii),

$$|\{v \in V : c_v^0 = 0\}| = \sum_{k=1}^i |V_k| + \frac{1}{2} \sum_{k=i+1}^m |V_k|$$

$$= (\sum_{k=i+1}^j |V_k|) - 1 + \frac{1}{2} \sum_{k=j+1}^m |V_k|$$

$$= |\{v \in V : c_v^0 = 1\}| - 1.$$

$$\text{sum}(c^0) = (n + 1)/2.$$
Thus, $\text{sum}(d^0) = (n + 1)/2$ and $\text{maj}(d^0) = 1$.

If $v$ is a master $c_v^1 = \text{maj}(d^0) = 1 = 1 - c_v^0$ (the last equality holds because \(\{v\} = H_k\) for some $k$ and $k \leq i$ by (i)). If $v$ is not a master, then $v \in H_k$ for some $k \leq m$ such that $|H_k| > 2$. If $k \leq j$, then $c_v^1 = 1 - c_v^0$ by Lemma 4. If $k > j$ then $c_v^1 = c_v^0$ by Lemma 5. Therefore, $c_v^1 = 1 - c_v^0$ if $v \in V_1 \cup \ldots \cup V_j$ and $c_v^1 = c_v^0$ if $v \in V_{j+1} \cup \ldots \cup V_m$. Thus,

$$
\left| \{v \in V : c_v^1 = 1 \} \right| = \sum_{k=1}^{i} |V_k| + \frac{1}{2} \sum_{k=j+1}^{m} |V_k| \\
= (\sum_{k=i+1}^{j} |V_k|) - 1 + \frac{1}{2} \sum_{k=j+1}^{m} |V_k| \\
= \left| \{v \in V : c_v^1 = 0 \} \right| - 1.
$$

So, $\text{maj}(c^1) = 0 \neq \text{maj}(c^0)$ and $G$ is not a majority consensus computer by Proposition 1.

For any $k = 0, 1, \ldots, (n-1)/2$, it is straightforward to construct a $G$ with $k$ masters satisfying conditions of Lemma 6. For example, if $k = 0$, take $G$ such that connected components of $G'$ are $P_{(n-1)/2}$ and $P_{(m+1)/2}$. If $k > 0$, $G$ whose connected components are its masters, $P_{k+1}$ and $C_{n-2k-1}$ is such example. Thus, there exist $G$ with $\delta(G) \geq n-3$ which are not m.c.c. and having exactly $k$ masters for every $k < (n+1)/2$. (Recall that, by Proposition 3, every $G$ with at least \((n+1)/2\) masters is a m.c.c.)

In order to prove that the Master Conjecture holds in the case $\delta(G) \geq n-3$, we need yet another lemma. In what follows we will say that $v_1, v_2, \ldots, v_k$ form a path $P_k$ if $v_i$ is adjacent to $v_{i+1}$ in $P_k$ for $i = 1, \ldots, (k-1)$. Similarly, we will say that $v_1, \ldots, v_k$ form a cycle $C_k$ if $v_1, \ldots, v_k$ form a path $P_k \subseteq C_k$ and $v_1$ is adjacent to $v_k$ in $C_k$.

**Lemma 7.** Let $c'$ be a coloring of $G$, $\delta(G) \geq n-3$, such that $\text{sum}(c') = (n + 1)/2$. Let $v_1, v_2, \ldots, v_k$ form $H \subseteq G'$, a connected component in $G'$ on $k \geq 3$ vertices. Suppose that there exists a $j < k/2$ such that $c'_{v_{i}} = i \mod 2$ for $i \leq 2j + 1$. If $2j + 1 < k$, also suppose that $c'_{v_{i}} = c'_{v_{i+2}}$ for $i > 2j + 1$.

Then $c'_{v_{i+1}} = c'_{v_{i}}$ for $i \leq 2j + 1$ and $c'_{v_{i+1}} = 1 - c'_{v_{i}}$ for $i > 2j + 1$.

**Proof.** Since $\delta(G) \geq n-3$, $H$ is a path or a cycle. Using (b) and (c) of Lemma 2, observe that $c'_{v_{i+1}} = c'_{v_{i}}$ for $i \leq 2j + 1$ (since each $v_i$ such that $c'_{v_{i}} = 0$ has both non-neighbors of color 1, while each $v_i$ such that $c'_{v_{i}} = 1$ has at least one non-neighbor of color 0) and that $c'_{v_{i+1}} = 1 - c'_{v_{i}}$ for $i > 2j + 1$ (if $c'_{v_{2j+2}} = \ldots = c'_{v_{k}} = 0$, then each such $v_i$ has a non-neighbor of color 0; if $c'_{v_{2j+2}} = \ldots = c'_{v_{k}} = 1$, then each such $v_i$ has all non-neighbors of color 1 because $c'_{v_{1}} = c'_{v_{2j+1}} = 1$).

**Theorem 5.** Let $G$ such that $\delta(G) \geq n-3$. If $G$ is a majority consensus computer, then $G$ contains a master.

**Proof.** Suppose $G$ does not contain a master. We’ll show that $G$ is not a majority consensus computer. Let $H_1 = (V_1, E'_1), H_2 = (V_2, E'_2), \ldots, H_m = (V_m, E'_m)$ be the connected components of $G'$. Since $G$ does not contain a master, $|V_l| \geq 2$, $l = 1, \ldots, m$. Choose an index $i$ such that $|V_i| + \ldots + |V_i| \leq (n-1)/2 < |V_i| + \ldots + |V_i| + |V_{i+1}|$. 

\[ |V_i| + \ldots + |V_i| \leq (n-1)/2 < |V_i| + \ldots + |V_i| + |V_{i+1}|. \]
If \(|V_i| + \ldots + |V_j| = (n-1)/2\), then the conditions of Lemma 6 are satisfied with \(i\), and with \(j = m\). Therefore, in this case, \(G\) is not a majority consensus computer.

For the rest of the proof we may assume that \(|V_i| + \ldots + |V_j| < (n-1)/2\). We may also assume that \(|V_i| + \ldots + |V_{i+1}| + |V_j| + (|V_{i+1}|/2) > (n-1)/2\). (If not, then \(|V_{i+1}|/2 + |V_{i+2}| + |V_{i+3}| + \ldots + (|V_m|) > (n-1)/2\) and we could map \(I\) to \(m+1-I\), i.e. \(H_I\) becomes \(H_{m+1-I}, I = 1, \ldots, m\).) Note that these imply that \(|V_j| \geq 3\).

Let \(v_1, v_2, \ldots, v_k\) form \(H_{i+1}\) and let

\[
j = (n-1)/2 - (|V_i| + \ldots + |V_{i-1}| + |V_i|) \tag{6}
\]

Note that \(j < k/2\). Set

\[
c_v^0 = \begin{cases} 
0 & v \in V_1 \cup V_2 \cup \ldots \cup V_i \\
i \mod 2 & v_i, \quad i = 1, \ldots, 2j+1 \\
1 & v_i, \quad i = 2j+2, \ldots, k \\
1 & v \in V_{i+2} \cup V_{i+3} \cup \ldots \cup V_m
\end{cases}
\]

Note that \(\text{sum}(c^0) = (n+1)/2\). By Lemma 4, \(c_v^1 = 1 - c_v^0\) for every \(v \notin V_{i+1}\). By Lemma 7, \(c_{v_i}^1 = 1 - c_{v_i}^0\) for \(i = 2j+2, \ldots, k\) and \(c_{v_i}^1 = c_{v_i}^0\) for \(i = 1, \ldots, 2j+1\). Thus, only \(j\) vertices colored by 0 and only \(j+1\) vertices colored by 1 do not switch color. Hence, \(\text{sum}(c^1) = |V_i| + \ldots + |V_j| + (j+1) = (n-1)/2+1 = (n+1)/2\) (the second equality follows from (6)).

Repeating the same argument for

\[
c_v^1 = \begin{cases} 
1 & v \in V_1 \cup V_2 \cup \ldots \cup V_i \\
i \mod 2 & v_i, \quad i = 1, \ldots, 2j+1 \\
0 & v_i, \quad i = 2j+2, \ldots, k \\
0 & v \in V_{i+2} \cup V_{i+3} \cup \ldots \cup V_m
\end{cases}
\]

we conclude that \(c^2 = c^0\). Thus, \(c^0, c^1, c^2, \ldots\) has period two. Therefore, \(G\) is not a majority consensus computer.

\[\square\]

5 Generalizations and Conclusions.

The main result of this paper is that failure-free computation of majority consensus by iterative applications of the local majority rule is possible only in the networks that are nowhere truly local (Theorem 4). In other words, the idea of solving a truly global task (reaching consensus on majority) by means of truly local computation only (local majority rule) is doomed for failure. However, even well connected networks of agents that are nowhere truly local might fail to reach majority consensus when iteratively applying the local majority rule. We have investigated the properties of majority consensus computers, i.e., the networks in which iterative application of the local majority rule always yields consensus in the initial majority state.
There are several generalizations and relaxations of the local majority process that one might consider more realistic and applicable than the process we study. Our results readily extend to many such generalizations and relaxations. For example, an obvious generalization of the model would be to allow weights on the edges and conduct the weighted local majority vote at each vertex at each time step according to those weights. If such weights on $E(G)$ are nonnegative real numbers, it is easy to see that there exists a multi-graph $M(G)$ on which our original process $LMP$ mimics this weighted generalization of $LMP$ on $G$. Here we only show one further result\(^6\) that emphasizes our point that understanding $LMP$ is fundamental to understanding any generalization of this process.

A simple generalization of the local majority process would allow vertex $v$ to have some resistivity towards color switch. Formally, for a nonnegative integer $k_v$, we define a $k_v$-local majority rule for vertex $v$:

$$
c^{t+1}_v = \begin{cases} 
c^{t}_v & \text{if } \{|w \in N_v : c^{t}_w = c^{t}_v\} \geq \frac{|N(v)|}{2} + k_v \\
1 - c^{t}_v & \text{if } \{|w \in N_v : c^{t}_w \neq c^{t}_v\} > \frac{|N(v)|}{2} + k_v
\end{cases}
$$

The value $k_v$ is called the resistivity value of vertex $v$ and we call the graph $G = (V,E)$ together with the set of vertex resistivities $\{k_v : v \in V\}$ a varied-resistivity graph. Similarly, the process defined by (7) is called the local majority process with resistivities. Note that the local majority process with resistivities where $k_v = 0, v \in V$, is exactly the local majority process.

**Theorem 6.** Let $G(V, E)$ be a varied-resistivity graph, with the vertex set $V = \{v_1, \ldots, v_n\}$, and the corresponding resistivities $R = \{k_{v_1}, \ldots, k_{v_n}\}$. The local majority process with resistivities on the varied-resistivity graph $G$ can be simulated by the local majority process on some graph $G'(V', E')$.

Apart from generalizations of the model, there are several directions that might be of potential interest. One such direction is to determine the complexity of the decision problem:

**MCC.** Input is a finite graph $G$. Is $G$ a majority consensus computer? Clearly, MCC is in co-NP because of Theorem 3 and it is very likely that MCC is co-NP complete. In [MP00], we solve this question by giving a complete characterization for graphs with $\delta(G) \geq (n-3)$, and a polynomial time algorithm that decides the problem.

We conjecture that every majority consensus computer $G$ contains a master, i.e., there exists $v \in V(G)$ such that $d(v) = |V(G)| - 1$ (see Master Conjecture in Section 3). We have proved that this conjecture holds for almost complete networks, i.e., networks that are in a way most natural candidates for a counterexample to the conjecture (Theorem 5). However, the Master Conjecture remains open.

\(^6\) See [MP00] for detailed discussion.
References


