Theorems of Carathéodory, Helly, and Tverberg without dimension

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Abstract

Motivated by Barman [6], we initiate a systematic study of the 'no-dimensional' analogues of some basic theorems in combinatorial and convex geometry, including the colorful Carathéodory's theorem, Tverberg's theorem, Helly's theorem as well as their fractional and colorful extensions.

1 Introduction

Carathéodory's classical theorem [8] from 1907 says that every point in the convex hull of a point set $P \subset \mathbb{R}^d$ is in the convex hull of a subset $Q \subset P$ with at most $d+1$ points. Can one require here that $|Q| \leq r$ for some fixed $r \leq d$? The answer is obviously 'no': when P is finite, the union of the convex hull of all r-element subsets of P, for $r \leq d$, has measure zero while the convex-hull of P, denoted by $\operatorname{conv} P$, may have positive measure. So one should set a more modest target. One way for this is to try to find, given $a \in \operatorname{conv} P$, a subset $Q \subset P$ with $|Q| \leq r$ so that a is close to $\operatorname{conv} Q$. Various versions of such a statement have been studied and proven before in different contexts—e.g., it is possible to find, given a parameter $\epsilon > 0$, $O\left(\frac{1}{\epsilon^2}\right)$ points of P whose convex-hull is within distance $\epsilon \cdot \operatorname{diam} P$ from a, where $\operatorname{diam} P$ denotes the diameter of P.

Recently Barman [6] found a beautiful connection of such a statement to additive approximation algorithms. The basic idea is the following. Consider an optimization problem that can be written as a bilinear program—namely maximizing/minimizing an objective function of the form $x^T Ay$, where the variables are $x, y \in \mathbb{R}^n$. If one knew the optimal value of the vector y, then the above bilinear program reduces to a linear one, which can be solved in polynomial time. Barman showed that several problems—among them computing Nash equilibria and densest bipartite subgraph problem—have two additional properties: *i)* y lies inside the convex-hull of some polytope, and *ii)* if y and y' are two close points in $\mathbb{R}^n$, then the value of the bilinear programs on y and y' are also close. Then applying the above approximate version of Carathéodory's theorem for the optimal point y (whose actual value we don't know), there must exist a point y', depending on a $O(\frac{1}{\epsilon^2})$-sized subset of the input, such that the distance between y and y' is small. Now one can enumerate all $O(\frac{1}{\epsilon^2})$-sized subsets to compute all such y', and thus arrive at an approximation to the bilinear program.

2 Our Results

Barman makes the first attempt to generalize this machinery for other convex and discrete geometry theorems to analogues of these theorems where, at the cost of additive approximations, the size of geometric structures is made *independent* of the dimension. The goal and motivation of our work is to initiate a systematic study of this.

We collectively name these analogues as 'no-dimension' theorems.

2.1 No-Dimensions Carathéodory's Theorem.

The colorful version of Carathéodory's theorem [3] states that if $a \in \bigcap_{i=1}^{d+1} \operatorname{conv} P_i$ where $P_i \subset \mathbb{R}^d$, then there is a transversal $T = \{p_1, \ldots, p_{d+1}\}$ such that $a \in \operatorname{conv} T$. Here a *transversal* of the set system $P_1, \ldots, P_{d+1}$ is a set $T = \{p_1, \ldots, p_{d+1}\}$ such that $p_i \in P_i$ for all $i \in [d+1]$. We extend this optimally—with the precise multiplicative constant—to the no-dimension case as follows.

THEOREM 2.1. *Let $P_1, \ldots, P_r$ be $r \geq 2$ point sets in $\mathbb{R}^d$ such that $a \in \bigcap_{i=1}^{r} \operatorname{conv} P_i$. Define $D = \max_{i \in [r]} \operatorname{diam} P_i$. Then there exists a transversal T such that*

$$d\left(a, \operatorname{conv} T\right) < \frac{D}{\sqrt{2r}}.$$

This bound is optimal (up to lower-order terms). Furthermore, such a transversal can be computed in deterministic $O(nd)$ time.

Remark. Barman [6] proves a qualitatively and quantitatively weaker statement, applicable only for the case $r = (d+1)$: given $(d+1)$ point sets $P_1, \ldots, P_{d+1}$, it is shown how to compute, using convex programming,

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a subset of r points P' with $|P' \cap P_i| \leq 1$ for each $i \in [d+1]$, such that $d(a, \operatorname{conv} P') = O\left(\frac{D}{\sqrt{r}}\right)$. We improve on this in two ways: a) the parameter r, the number of sets P_i, can be any value, in particular $r \leq d$, and thus truly does not depend on the dimension, and b) the running time of Barman's algorithm is $(nd)^{O(r)}$.

Remark. For the case $r = (d+1)$, finding the transversal T such that $a \in \operatorname{conv} T$ in time polynomial in the number of the input points and dimension is a longstanding open problem (see [19]). Barman's work implies an algorithm that computes an approximate transversal with running time $(nd)^{O(r)}$, while Theorem 2.1 improves the running time to $O(nd)$.

Remark. Colorful Carathéodory's theorem has been extended earlier [20] for the case $r = \lfloor \frac{d}{2} \rfloor + 1$, though in a different way.

By setting $P_1 = \ldots = P_r = P$ and applying Theorem 2.1, we recover the precise statement—optimal even to a multiplicative constant—of no-dimension Carathéodory's theorem.

THEOREM 2.2. *Let P be a set of n points in $\mathbb{R}^d$ and $r \in [n]$. Then for any point $a \in \operatorname{conv} P$, there exists a subset Q of P with $|Q| = r$ such that*

$$d(a, \operatorname{conv} Q) < \frac{\operatorname{diam} P}{\sqrt{2r}}. \tag{2.1}$$

This bound is optimal (up to lower-order terms). Furthermore, such a subset Q can be found in polynomial time.

Remark. This bound is the best possible: when $d = n-1$ and P is the set of vertices of a regular $(n-1)$-dimensional simplex whose centre is a, then for every $Q \subset P$ with $|Q| = r$,

$$d(a, \operatorname{conv} Q) = \sqrt{\frac{1}{2r} - \frac{1}{2n}} \operatorname{diam} P.$$

Remark. There is cone version of Carathéodory's theorem which is stronger than the convex version. Writing $\operatorname{pos} P$ for the cone hull of $P \subset \mathbb{R}^d$, it says the following. Assume $P \subset \mathbb{R}^d$ and $a \in \operatorname{pos} P$ and $a \neq o$. Then there is $Q \subset P$ with $|Q| \leq d$ such that $a \in \operatorname{pos} Q$. The corresponding no-dimension variant would say that, under the same condition and given $r < d$, there is $Q \subset P$ with $|Q| \leq r$ such that the angle between a and the cone $\operatorname{pos} Q$ is smaller then some function of r that goes to zero as $r \to \infty$. Unfortunately, this is not true as the following example shows.

Let $P = \{v_1, \ldots, v_d\}$ be the set of vertices of a regular $(d-1)$-dimensional simplex. Assume that its centre of gravity, a, is the closest point of $\operatorname{conv} P$ to the origin, and $|a| = h$ is small. Then $a \in \operatorname{pos} P$. For any subset Q of P, of size $r < d$, $\operatorname{pos} Q$ is contained in the boundary of the cone $\operatorname{pos} P$. The minimal angle ϕ between a and a vector on the boundary of $\operatorname{pos} Q$ satisfies

$$\tan \phi = \frac{|v_1 - a|}{(d-1)h},$$

which can be made arbitrarily large by choosing h small enough.

A strengthening of the colorful Carathéodory's theorem from [2] and [15] states that given non-empty sets $P_1, \ldots, P_{d+1} \in \mathbb{R}^d$ such that $a \in \operatorname{conv}(P_i \cup P_j)$ for every $i, j \in [d+1], i \neq j$, there is a transversal $T = \{p_1, \ldots, p_{d+1}\}$ such that $a \in \operatorname{conv} T$. It is shown in [2] that the "union of any two" condition here cannot be replaced by the "union of any three" (or more) condition. We extend such a result to the no-dimensional case.

THEOREM 2.3. *Let $P_1, \ldots, P_r$ be $r \geq 2$ point sets in $\mathbb{R}^d$, $D = \max_{i \in [r]} \operatorname{diam} P_i$, and $t \in [r-1]$. Assume that for distinct $i_1, i_2, \ldots, i_t \in [r]$, $a \in \operatorname{conv}(P_{i_1} \cup \ldots \cup P_{i_t})$. Then there exists a transversal $T = \{p_1, \ldots, p_r\}$, such that*

$$d(a, \operatorname{conv} T) \leq \beta \cdot \frac{D}{\sqrt{r-t+1}},$$

where $\beta = 2\sqrt{\frac{\ln 4}{3}} = 1.3595\ldots$

Remark. This proof is a routine application of the Frank-Wolfe procedure (e.g., see [11]). For the case $t = 1$, it implies a slightly weaker bound than Theorem 2.1, i.e., it finds a transversal T with

$$d(a, \operatorname{conv} T) \leq \beta \cdot \frac{D}{\sqrt{r}}. \tag{2.2}$$

2.2 No-Dimensions Tverberg's Theorem. We next prove the no-dimension version of Tverberg's famous theorem [24].

THEOREM 2.4. *Given a set P of n points in $\mathbb{R}^d$ and an integer $k \in [n]$, there exists a point $q \in \mathbb{R}^d$ and a partition of P into k sets $P_1, \ldots, P_k$ such that*

$$d(q, \operatorname{conv} P_i) \leq \left(2 + \sqrt{2}\right) \cdot \sqrt{\frac{k}{n}} \operatorname{diam} P \text{ for every } i \in [k].$$

Apart from the constant $(2 + \sqrt{2})$, this bound is tight.

Actually we will prove the no-dimension version of the more general colored Tverberg Theorem (see [25] and [7]).

THEOREM 2.5. *Let $C_1, \ldots, C_r \subset \mathbb{R}^d$ be r pairwise-disjoint sets of points, and with $|C_i| = k$ for all $i \in [r]$. Let $P = \bigcup_1^r C_j$. Then there is a point $q \in \mathbb{R}^d$ and a partition $P_1, \ldots, P_k$ of P such that $|P_i \cap C_j| = 1$ for every $i \in [k]$ and every $j \in [r]$ satisfying*

$$d\left(q, \operatorname{conv} P_i\right) \leq \gamma \cdot \frac{\operatorname{diam} P}{\sqrt{r}} \text{ for every } i \in [k],$$

with $\gamma = 1 + \sqrt{2}$. Apart from the constant γ, this bound is tight.

Remark. Theorem 2.5 implies Theorem 2.4: write $|P| = n = kr + s$ with $k \in \mathbf{N}$ so that $0 \leq s \leq k - 1$. Then delete s elements from P and split the remaining set into sets (colours) $C_1, \ldots, C_r$, each of size k. Apply the coloured version and add back the deleted elements (anywhere you like). The outcome is the required partition; the extra factor $\sqrt{2}$ between the constants $2 + \sqrt{2}$ and $1 + \sqrt{2}$ comes when $k = 2$ and $2r$ is only slightly smaller than n. But Theorem 2.4 holds with constant $1 + \sqrt{2}$ (instead of $2 + \sqrt{2}$) when r divides n.

Remark. The bounds given in Theorems 2.4 and 2.5 are best possible apart from the constant γ. Indeed, the regular simplex with $n = kr$ vertices shows, after a fairly direct computation, that for every point $q \in \mathbb{R}^{n-1}$ and every partition $P_1, \ldots, P_k$ of the vertices

$$\max_{i \in [k]} d(q, \operatorname{conv} P_i) \geq \frac{\operatorname{diam} P}{\sqrt{2r}} \sqrt{\frac{k-1}{k} \frac{n-1}{n-2}}.$$

The computation is simpler in the colored case. We omit the details.

2.3 No-Dimensions Helly's Theorem. In the same spirit there is the no-dimension Helly's theorem. In fact we first give a more general theorem. Let $B(b, \rho)$ denote the Euclidean ball centered at $b \in \mathbb{R}^d$ of radius ρ.

THEOREM 2.6. *Let $\alpha \in (0, 1]$, $\beta = 1 - (1 - \alpha)^{\frac{1}{k}}$ and $\rho \geq 0$. Let $\mathcal{F}$ be a finite and nonempty family of convex sets in $\mathbb{R}^d$. Assume that for an α-fraction of k-tuples $K_1, \ldots, K_k$ of $\mathcal{F}$, the set $\bigcap_{i=1}^k K_i$ has a point in $B(b, \rho)$. Then there is $q \in \mathbb{R}^d$ such that at least $\beta|\mathcal{F}|$ elements of $\mathcal{F}$ intersect the ball $B\left(q, \frac{\rho}{\sqrt{k}}\right)$.*

Setting $\rho = 1$, $\alpha = 1$ yields the following no-dimensions Helly's theorem.

THEOREM 2.7. *Assume $K_1, \ldots, K_n$ are convex sets in $\mathbb{R}^d$ and $k \in [n]$. For $J \subset [n]$ define $K(J) = \bigcap_{j \in J} K_j$. If the Euclidean unit ball $B(b, 1)$ centered at $b \in \mathbb{R}^d$ intersects $K(J)$ for every $J \subset [n]$ with $|J| = k$, then there is point $q \in \mathbb{R}^d$ such that*

$$d(q, K_i) \leq \frac{1}{\sqrt{k}} \text{ for all } i \in [n].$$

Remark. A more precise inequality is $d(q, K_i) \leq \sqrt{\frac{n-k}{k(n-1)}}$ which is best possible, but more technical to establish.

We extend the result to the colorful version of Helly's theorem, which is due to Lovász (see [3]). A *transversal* $\mathcal{T}$ of the system $\mathcal{F}_1, \ldots, \mathcal{F}_k$ of k families of objects in $\mathbb{R}^d$ is $\mathcal{T} = \{K_1, \ldots, K_k\}$ where $K_i \in \mathcal{F}_i$ for all $i \in [k]$. We define $K(\mathcal{T}) = \bigcap_1^k K_i$.

THEOREM 2.8. *Let $\mathcal{F}_1, \ldots, \mathcal{F}_k$ be finite and non-empty families of convex sets in $\mathbb{R}^d$. Assume that for every $p \in \mathbb{R}^d$ and all $i \in [k]$, there are at least m_i sets $K \in \mathcal{F}_i$ with $d(p, K) > \rho_i$. Then for every $q \in \mathbb{R}^d$ there are at least $\prod_1^k m_i$ transversals $\mathcal{T}$ such that*

$$d\left(q, K\left(\mathcal{T}\right)\right) > \sqrt{\rho_1^2 + \ldots + \rho_k^2},$$

with the convention that $d(q, \emptyset) = \infty$.

Remark. When $\rho_i = 0$, $m_i = 1$ for all $i \in [k]$, then the sets in $\mathcal{F}_i$ have no point in common, and when $\mathcal{F} = \mathcal{F}_1 = \ldots = \mathcal{F}_k$ and $k = d + 1$, the conclusion is that for every point q there are some $d + 1$ (or fewer) sets in $\mathcal{F}$ whose intersection does not contain q. This implies, via an additional extremal argument, Helly's original theorem. In the same way, when $k = d + 1$, $\rho_i = 0$ and $m_i = 1$ for every $i \in [k]$, we get the colorful version of Helly's theorem.

Remark. We mention that the value $\prod_1^k m_i$ is best possible as shown by the following example. Let $e_1, \ldots, e_k$ denote the standard basis vectors of $\mathbb{R}^k$ and choose a real number r_i larger than ρ_i, but only slightly larger. Set $v_i = r_i e_i$. For every $i \in [k]$ the family $\mathcal{F}_i$ contains m_i copies of the hyperplane $H_i^- = \{x \in \mathbb{R}^k : v_i(x - v_i) = 0\}$ and also m_i copies of the hyperplane $H_i^+ = \{x \in \mathbb{R}^k : v_i(x + v_i) = 0\}$, and furthermore some finitely many copies of the whole space $\mathbb{R}^k$. It is clear that the smallest ball intersecting every set in $\mathcal{F}_i$ is $r_i B^k$. Moreover, given a transversal $H_1^{\epsilon_1}, \ldots, H_k^{\epsilon_k}$ of the system $\mathcal{F}_1, \ldots, \mathcal{F}_k$ with $\epsilon_i \in \{+, -\}$, their intersection is a point at distance $r = \sqrt{r_1^2 + \ldots + r_k^2}$ from the origin, and there are exactly $\prod_1^k m_i$ such transversals. All other transversals of the system have a point in the interior of rB, and then also in ρB if the r_is are chosen close enough to ρ_i.

Next is the no-dimension colorful variant of Helly's theorem.

THEOREM 2.9. *Let $\mathcal{F}_1, \ldots, \mathcal{F}_k$ be finite and non-empty families of convex sets in $\mathbb{R}^d$ such that for every transversal $\mathcal{T}$, the set $K(\mathcal{T})$ intersects the Euclidean unit ball $B(b,1)$. Then there is $i \in [k]$ and a point $q \in \mathbb{R}^d$ such that*

$$d(q,K) \leq \frac{1}{\sqrt{k}} \text{ for all } K \in \mathcal{F}_i.$$

Remark. The proof follows from Theorem 2.8 with $\rho_i = \frac{1}{\sqrt{k}}$.

We next give an application of no-dimension Helly's theorem—the no-dimensional centerpoint theorem.

THEOREM 2.10. *Let P be a set of n points in $\mathbb{R}^d$ lying in the unit ball $B(o,1)$. For any integer $k > 0$, there exists a point $q \in \mathbb{R}^d$ such that any closed half-space containing $B\left(q, \frac{1}{\sqrt{k}}\right)$ contains at least $\frac{n}{k}$ points of P.*

Remark. Given a set P of n points in $\mathbb{R}^d$, the centerpoint theorem states the existence of a point $q \in \mathbb{R}^d$ such that any half-space containing q contains at least $\frac{n}{(d+1)}$ points of P. The proportion $\frac{1}{(d+1)}$ cannot be improved, in the sense that there exist examples where every point in $\mathbb{R}^d$ has some half-space containing it and containing at most $\lceil \frac{n}{(d+1)} \rceil$ points of P. Our theorem goes beyond this at the cost of approximate inclusion.

2.4 No-Dimensions Piercing Theorems.

We also give a no-dimension version of the selection lemma [3] and [19] and the weak ϵ-net theorem [1].

THEOREM 2.11. *Given a set $P \subset \mathbb{R}^d$ with $|P| = n$ and $D = \operatorname{diam} P$ and an integer $r \in [n]$, there is a point $q \in \mathbb{R}^d$ such that the ball $B\left(q, \frac{3.5 \cdot D}{\sqrt{r}}\right)$ intersects the convex hull of $r^{-r}\binom{n}{r}$ r-tuples in P.*

As expected, the no-dimension selection lemma implies the no-dimension version of the weak ϵ-net theorem (see the survey [21]).

THEOREM 2.12. *Assume $P \subset \mathbb{R}^d$, $|P| = n$, $D = \operatorname{diam} P$, $r \in [n]$ and $\epsilon > 0$. Then there is a set $F \subset \mathbb{R}^d$ of size at most $r^r \epsilon^{-r}$ such that for every $Y \subset P$ with $|Y| \geq \epsilon n$,*

$$\left(F + \frac{3.5 \cdot D}{\sqrt{r}} B\right) \cap \operatorname{conv} Y \neq \emptyset.$$

3 Previous work

Results similar to Theorem 2.2 have been known for some time, many motivated by geometric measure concentration and Banach space theory. The first one seems to be due to B. Maurey and appeared in 1981 in a paper by Pisier [22]. It says that if a set P lies in the unit ball of the space, $a \in \operatorname{conv} P$ and $r \in \mathbf{N}$, then a is contained in a ball of radius $\frac{c}{\sqrt{r}}$ whose center is the centroid of multiset $Q \subset P$ with exactly r elements, where c is a constant. The proof uses Khintchin's inequality and is probabilistic. Further results of this type were proved by Carl [9] and by Carl and Pajor [10] and used in geometric Banach space theory. The underlying space is not necessarily the Euclidean, for instance L_p spaces are allowed. Essentially the same result (with an almost identical proof) was discovered in 2015 independently by S. Barman [6]. The motivation there is different: determination of approximate Nash equilibria in two-person games, and approximation algorithms for the densest k-subgraph problem. In Barman's result the probabilistic proof is turned into a randomized algorithm. Some of the results in this area have become highly influential in geometric concentration of measure (see [13, 12] for an overview) as for instance Talagrand's inequality (convex subsets of the cube of some measure are highly exhaustive) which is dimension independent as well.

A similar inequality was proved by Bárány and Füredi [4] in 1987 with a very different purpose. They showed that every deterministic polynomial time algorithm that wants to compute the volume of a convex body in $\mathbb{R}^d$ has to make a huge error, namely, a multiplicative error of order $\left(\frac{d}{\log d}\right)^d$. Their proof is based on a lemma, similar to Theorem 2.2. Before stating it we have to explain what the ρ-*cylinder* above a set $Q \subset \mathbb{R}^d$ is where $|Q| \leq d$. Let B denote the Euclidean unit ball of $\mathbb{R}^d$, and let L be the linear (complementary) subspace orthogonal to the affine hull of Q. Then the cylinder in question is $Q^\rho := (L \cap \rho B) + \operatorname{conv} Q$. With this notation the key lemma in [4] says that given $P \subset B$ and $r \leq d$, every point in $\operatorname{conv} P$ is contained in a cylinder $Q^{\rho(d,r)}$ for some $Q \subset P$ of size r, here $\rho(d,r) = \sqrt{\frac{d-r+1}{d(r-1)}}$. This becomes $\frac{1}{\sqrt{r-1}}$ in the no dimension setting as

$$\sqrt{\frac{d-r+1}{d(r-1)}} = \sqrt{\frac{1}{r-1} - \frac{1}{d}} < \sqrt{\frac{1}{r-1}}.$$

and would give, in Theorem 2.2 the estimate

$$d\left(a, \operatorname{conv} Q\right) \leq \frac{R}{\sqrt{r-1}}$$

where R is the radius of the ball circumscribed to P. By Jung's theorem [16], $R \leq \sqrt{\frac{n-1}{2n}} D$ which gives in our setting the slightly weaker upper bound

$$d\left(a, \operatorname{conv} Q\right) \leq \frac{D}{\sqrt{2(r-1)}}.$$

The proof of the lemma from [4] does not seem to extend to the case of Theorem 2.1.

We note that the estimates in Maurey's lemma (and Barman's), and the one in [4], and also in Theorem 2.2 and 2.1 are all of order $\frac{1}{\sqrt{r}}$ but the constants are different. Part of the reason is that the setting is slightly different: in the first ones P is a subset of the unit ball of the space while in Theorem 2.2 and 2.1 (and elsewhere in this paper) the scaling parameter is $\operatorname{diam} P$.

4 Proof of Theorem 2.1

Proof. Given a finite set $Q \subseteq \mathbb{R}^d$, denote by $c(Q)$ the centroid of Q, that is,

$$c(Q) = \frac{1}{|Q|}\sum_{x\in Q} x.$$

First we prove the theorem in a special case, namely when $a = c(P_i)$ for every $i \in [r]$. Set $n_i = |P_i|$.

We can assume (after a translation if necessary) that $a = o$. We compute the average, denoted Ave, of

$$c(Q)\,c(Q) = c(Q)^2 = \left(\frac{1}{r}\sum_{x\in Q} x\right)^2 = \frac{1}{r^2}\left(\sum_{x\in Q} x\right)^2,$$

taken over all transversals Q of the sets $P_1, \dots, P_r$. Here $\left(\sum_{x\in Q} x\right)^2$ is a linear combination of terms of the form x^2 ($x \in P_i$), $i \in [r]$ and $2xy$ ($x \in P_i$, $y \in P_j$, $i, j \in [r]$, $i < j$).

By symmetry, in Ave $\left(\sum_{x\in Q} x\right)^2$, the coefficient of each x^2 with $x \in P_i$ is the same and is equal to $1/n_i$. Similarly, the coefficient of each $2xy$ with $x \in P_i$, $y \in P_j$ is the same and is equal to $1/(n_i n_j)$. This follows from the fact that in every $\left(\sum_{x\in Q} x\right)^2$, out of all $x \in P_i$, exactly one x^2 appears. Similarly, out of all pairs $x \in P_i, y \in P_j$, exactly one $2xy$ appears. So we have

$$\begin{aligned}
\text{Ave }\left[c(Q)^2\right] &= \text{Ave }\left(\frac{1}{r}\sum_{x\in Q} x\right)^2 \\
&= \frac{1}{r^2}\left[\sum_1^r \frac{1}{n_i}\sum_{x\in P_i} x^2 + \sum_{1\le i<j\le r}\ \sum_{x\in P_i,\, y\in P_j} \frac{1}{n_i n_j} 2xy\right] \\
&= \frac{1}{r^2}\left[\sum_1^r \frac{1}{n_i}\sum_{x\in P_i} x^2 + 2\sum_{1\le i<j\le r}\left(\frac{1}{n_i}\sum_{x\in P_i} x\right)\left(\frac{1}{n_j}\sum_{y\in P_j} y\right)\right] \\
&= \frac{1}{r^2}\left[\sum_1^r \frac{1}{n_i}\sum_{x\in P_i} x^2 + 2\sum_{1\le i<j\le r} c(P_i)c(P_j)\right] \\
&= \frac{1}{r^2}\sum_1^r \frac{1}{n_i}\sum_{x\in P_i} x^2 \le \frac{1}{r^2}\sum_1^r D^2 = \frac{D^2}{r},
\end{aligned}$$

which is slightly weaker than our target. We need a simple (and well-known) lemma.

LEMMA 4.1. *Assume $X \subset \mathbb{R}^d$, $|X| = n$, $\sum_{x\in X} x = o$ and $D = \operatorname{diam} X$. Then*

$$\frac{1}{n}\sum_{x\in X} x^2 \le \frac{n-1}{2n}D^2 < \frac{D^2}{2}.$$

Proof. For distinct $x, y \in X$ we have $d(x,y)^2 = (x-y)(x-y) = x^2 + y^2 - 2xy \le D^2$. Furthermore, $\sum_{x\in X} x = o$ implies that $\sum_{x\in X} x^2 = -\sum_{\substack{x,y\in X\\ x\ne y}} 2xy$ with the last sum taken over all distinct $x, y \in X$. Thus

$$\begin{aligned}
\sum_{x\in X} x^2 &= -\sum_{\substack{x,y\in X\\ x\ne y}} 2xy \le \sum_{\substack{x,y\in X\\ x\ne y}}\left(D^2 - x^2 - y^2\right) \\
&= \binom{n}{2}D^2 - (n-1)\sum_{x\in X} x^2,
\end{aligned}$$

implying the statement.

Using this for estimating $\text{Ave}\, c(Q)^2$ we get

$$\text{Ave}\, c(Q)^2 = \frac{1}{r^2}\sum_1^r \frac{1}{n_i}\sum_{x\in P_i} x^2 < \frac{1}{r^2}\sum_1^r \frac{D^2}{2} = \frac{D^2}{2r}.$$

This shows that there is a transversal T with $|c(T)| < D/\sqrt{2r}$. Then $d(o, \operatorname{conv} T) < D/\sqrt{2r}$, which proves the theorem in the special case when each $c(P_i) = a$.

In the general case a is a convex combination of the elements in P_i for every $i \in [r]$, that is,

(4.3)
$$a = \sum_{x\in P_i} \alpha_i(x)x, \text{ with } \alpha_i(x) \ge 0 \text{ and } \sum_{x\in P_i} \alpha_i(x) = 1$$

By continuity it suffices to prove the statement when all $\alpha_i(x)$ are rational. Assume $\alpha_i(x) = \frac{m_i(x)}{m_i}$ where $m_i(x)$ is a non-negative integer and $m_i = \sum_{x\in P_i} m_i(x) > 0$.

Now let P_i^* be the multiset containing $m_i(x)$ copies of every $x \in P_i$, with $|P_i^*| = m_i$. Now note that for each $i \in [r]$,

$$c\left(P_i^*\right) = \frac{1}{m_i}\sum_{x\in P_i^*} x = \frac{1}{m_i}\sum_{x\in P} m_i(x)x = a,$$

and $D = \max \operatorname{diam} P_i^*$. The previous argument applies to the sets $P_1^*, \ldots, P_r^*$, and gives a transversal $T^* = \{p_1, \ldots, p_r\}$ of the system $P_1^*, \ldots, P_r^*$ such that

$$d(a, \operatorname{conv} T^*) < \frac{D}{\sqrt{2r}}.$$

To complete the proof we note that $T = T^*$ is a transversal of the system $P_1, \ldots, P_r$ as well.

Equivalent formulation. One can express this proof in the following way. Choose the point $x_i \in P_i$ randomly, independently, with probability $\alpha_i(x)$ for all $i \in [r]$ where $\alpha_i(x)$ comes from (4.3). This gives the transversal $\{x_1, \ldots, x_r\}$. We set again $a = o$. The expectation of $\left(\frac{1}{r}(x_1 + \ldots + x_r)\right)^2$ turns out to be

$$\frac{D^2}{2r}\left(1 - \frac{1}{r}\sum \frac{1}{n_i}\right) < \frac{D^2}{2r}.$$

The computations are similar and this proof may be somewhat simpler than the original one. But the original one is developed further in the proof of Theorem 2.5. Actually, the random part of the proofs in [22], [9], [10], [6] are essentially the same except that they don't use the product distribution $\prod_1^r \alpha_i(\cdot)$ just the r-fold product of $\alpha(\cdot)$ coming from the convex combination $a = \sum_{x \in P} \alpha(x)x$.

We now present the deterministic algorithm, derived by derandomizing the above proof. We state it for the case assuming that $\alpha_i(x) = \frac{1}{n_i}$ for each $i \in [r]$ and $x \in P_i$; this is the case when $a = c(P_i)$ for all $i \in [r]$; the general case follows in the same way, by derandomizing the probabilistic proof that picks each $x \in P_i$ with probability $\alpha_i(x)$ (as outlined in the equivalent formulation above).

We will iteratively choose the points in the sets. Assume we have selected the points $f_i \in P_i$ for $i = s+1, \ldots, r$. We also need to be able to evaluate the conditional expectation $\mathbb{E}\left[c\left(\{x_1, \ldots, x_s, f_{s+1}, \ldots, f_r\}\right)^2 \middle| f_{s+1}, \ldots, f_r\right]$, where the expectation is over the points $x_1, \ldots, x_s$ chosen uniformly from the sets $P_1, \ldots, P_s$. This can be done, as

$$\begin{aligned} &\operatorname{Ave}\left(\frac{1}{r}\left(\sum_{i=1}^s x_i + \sum_{i=s+1}^r f_i\right)\right)^2 \\ = \ &\frac{1}{r^2}\Bigg[\operatorname{Ave}\left(\sum_{i=1}^s x_i\right)^2 + \\ &\left(\sum_{i=s+1}^r f_i\right)^2 + 2\left(\sum_{i=s+1}^r f_i\right)\operatorname{Ave}\left(\sum_{i=1}^s x_i\right)\Bigg]. \end{aligned}$$

Now, as shown earlier, we have

$$\operatorname{Ave}\left(\sum_{i=1}^s x_i\right)^2 = \sum_{i=1}^s \frac{1}{n_i}\sum_{x \in P_i} x^2. \tag{4.4}$$

Similarly, one can compute $\operatorname{Ave}\left(\sum_{i=1}^s x_i\right)$ exactly. Thus one can compute

$$\mathbb{E}\left[c\left(\{x_1, \ldots, x_s, f_{s+1}, \ldots, f_r\}\right)^2 \middle| f_{s+1}, \ldots, f_r\right]$$

exactly. We can pre-compute the postfix sums in equality (4.4) at the beginning of the algorithm, in total time $O\left(d\sum_i^r n_i\right)$. Then the above expectation can be computed in $O(1)$ time. Now, given the sets $P_1, \ldots, P_r$, one can try all possible points $x \in P_r$ to find the point $f_r \in P_r$ such that

$$\mathbb{E}\left[c\left(\{x_1, \ldots, x_{r-1}, f_r\}\right)^2\right] \le \mathbb{E}\left[c\left(\{x_1, \ldots, x_{r-1}, x_r\}\right)^2\right]$$

in time $O(n_r)$. This fixes the point f_r, and now reiterate to find the point f_{r-1}, and so on till we have fixed all the points $f_1, \ldots, f_r$ with the required upper-bound on $c(\cdot)^2$:

$$\begin{aligned} \mathbb{E}\left[c\left(\{f_1, \ldots, f_r\}\right)^2\right] &\le \mathbb{E}\left[c\left(\{x_1, f_2, \ldots, f_r\}\right)^2\right] \\ &\le \cdots \le \mathbb{E}\left[c\left(\{x_1, x_2, \ldots, x_r\}\right)^2\right] \\ &< \frac{D^2}{2r}. \end{aligned}$$

Overall, the running time is $O\left(d\sum_{i=1}^r |P_i|\right)$.

Remark. The proof of Theorem 2.1 also works when the sets $\operatorname{conv} P_i$ do not intersect but there is a point close to each. Recall that $B(a, \rho)$ denotes the Euclidean ball centered at $a \in \mathbb{R}^d$ of radius ρ.

Lemma 4.2. *Let $P_1, \ldots, P_r$ be $r \ge 2$ point sets in $\mathbb{R}^d$, $D = \max_{i \in [r]} \operatorname{diam} P_i$ and $\eta > 0$. Assume that there exists a point $a \in \mathbb{R}^d$ such that $B(a, \eta D) \cap \operatorname{conv} P_i \neq \emptyset$ for every $i \in [r]$. Then there exists a transversal T such that*

$$d(a, \operatorname{conv} T) \le \frac{D}{\sqrt{2r}}\sqrt{1 + 2(r-1)\eta^2}.$$

Proof. The proof works up to the point where $\sum c(P_i)c(P_j)$ appears. This time the sum is not zero but every term is at most $\eta^2 D^2$, and there are $\binom{r}{2}$ terms. This gives the required bound.

5 Proof of Theorem 2.3

Proof. The proof and its calculations are similar to other applications of the Frank-Wolfe method (see [11] and the references therein). For completeness, we present the proof in our setting.

By translation, we can assume that $a = o$. For simpler notation we write $|q| = d(o,q)$ when $q \in \mathbb{R}^d$, and the scalar product of vectors $u, v \in \mathbb{R}^d$ is written as uv.

Initially, pick an arbitrary point of P_1, say $p_1 \in P_1$. We are going to construct a sequence $i_1 = 1, i_2, \ldots, i_{r-t+1}$ consisting of $r - t + 1$ distinct integers with $i_j < j+t$ and a point $p_{i_j} \in P_{i_j}$ for each i_j as follows. We start with an arbitrary $p_1 \in P_1$ and set $q_1 = p_1$. Assume $i_1 = 1, i_2, \ldots, i_j$ have been constructed and set $P^j = \{p_{i_1}, \ldots, p_{i_j}\}$ and $I^j = \{i_1, \ldots, i_j\}$. Let q_j be the nearest point of $\operatorname{conv} P^j$ to the origin, define $v_j = o - q_j$ and let Q^j be the union of all P_i with $i \in [j+t] \setminus I^j$. Define

$$p = \arg\max_{x \in Q^j} (x - q_j)\, v_j.$$

Of course this point p belongs to some P_i with $i \in [j+t] \setminus I^j$; denote it by $P_{i_{j+1}}$ and set $p_{i_{j+1}} = p$. Let q_{j+1} be the nearest point of $\operatorname{conv} P^j \cup \{p_{i_{j+1}}\}$ to the origin.

Lemma 5.1.

$$|q_{j+1}| \le \left(1 - \frac{|q_j|^2}{2D^2}\right) \cdot |q_j|.$$

Proof. See Figure 1. In the triangle with vertices q_j, o, q_{j+1}, we have $\sin\theta = \frac{|q_{j+1}|}{|q_j|}$. On the other hand, in the triangle with vertices $q_j, p_{i_{j+1}}, p'$, we get

$$\cos\theta = \frac{|q_j - p'|}{|q_j - p_{i_{j+1}}|} \ge \frac{|q_j|}{D}.$$

This requires $|q_j - p'| \ge |q_j|$. Note that $o \in \operatorname{conv} Q^j$ due to the fact that Q^j is the union of at least t of the P_i's. Thus $|q_j - p'| \ge |q_j|$ since any half-space containing o must contain some point of Q^j. Using the fact that $\sin^2\theta + \cos^2\theta = 1$, we get

$$(5.5)\quad |q_{j+1}| \le \sqrt{1 - \left(\frac{|q_j|}{D}\right)^2} \cdot |q_j| \le \left(1 - \frac{|q_j|^2}{2D^2}\right) \cdot |q_j|$$

Lemma 5.1 implies that at the start, when $|q_j|$ is large, the decrease is correspondingly larger, and this slows down with more iterations. Specifically, for an integer $l \ge 1$, let k_l be the number of indices i_j such that

$$(5.6)\qquad \frac{D}{2^l} < |q_j| \le \frac{D}{2^{l-1}}.$$

Let $j' \in \{1, \ldots, r-t+1\}$ be the smallest index for which (5.6) is true. By Lemma 5.1 and the fact that

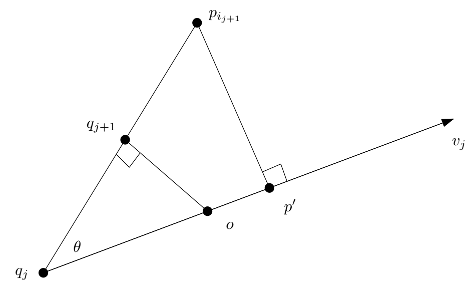

Figure 1: An iteration of the algorithm.

$|q_j|$ is a non-increasing function of j, we have

$$|q_{j'+k_l-1}| \le \prod_{m=j'}^{j'+k_l-1} \left(1 - \frac{|q_m|^2}{2D^2}\right) \cdot |q_{j'}| \le \left(e^{-\frac{|q_{j'+k_l-1}|^2}{2D^2}}\right)^{k_l} \cdot |q_{j'}|.$$

Now the fact that $|q_{j'+k_l-1}| > \frac{D}{2^l}$ implies that $k_l < 2^{2l} \cdot \ln 4$. Thus the maximum number of iterations i_j for which we have $|q_j| > \frac{D}{2^t}$ is at most

$$\sum_{l=1}^{t} 2^{2l} \cdot \ln 4 = \ln 4 \cdot \frac{4(4^t - 1)}{3}.$$

This is at most $r - t + 1$ for $2^t = \sqrt{\frac{3(r-t+1)}{4\ln 4} + 1}$. In other words, after $r - t + 1$ iterations, we have

$$|q_{r-t+1}| \le \frac{D}{\sqrt{\frac{3(r-t+1)}{4\ln 4} + 1}} \le 2\sqrt{\frac{\ln 4}{3}} \cdot \frac{D}{\sqrt{r-t+1}}.$$

We extend this partial transversal to a complete one with an arbitrary choice of $p_i \in P_i$ for all $i \in [r - t] \setminus I^{r-t+1}$. The new transversal satisfies the required inequality.

Remark. In this proof the first point $p_1 \in P_1$ can be chosen arbitrarily, even the condition $a \in \operatorname{conv} P_1$ is not needed. This implies that there are at least $|P_1|$ suitable transversals because the starting point p_1 can be chosen in $|P_1|$ different ways, and each gives a different transversal. We remark further that the proof is an algorithm that finds the transversal Q.

6 Proof of Theorem 2.5

We need a preliminary lemma. Recall that P is the disjoint union of sets (considered colors) $C_1, \ldots, C_r \subset \mathbb{R}^d$, and each C_j has size $k \ge 2$, the case $k = 1$ is trivial. Set $n = rk$.

Lemma 6.1. *Under the above conditions there is a subset $Q \subset P$ with $|Q \cap C_j| = \lfloor \frac{k}{2} \rfloor$ for every $j \in [r]$ such that*

(i) $d(c(Q), c(P)) \leq \sqrt{\frac{k}{2(k-1)n}} \operatorname{diam} P$ *if k is even,*

(ii) $d(c(Q), c(P)) \leq \sqrt{\frac{(k-2)(k+1)}{2(k-1)^2 n}} \operatorname{diam} P$ *if k is odd.*

Proof. Assume again that $c(P) = o$ and write $D = \operatorname{diam} P$. We use an averaging argument again, this time averaging over all subsets Q of P with $|Q \cap C_j| = \lfloor \frac{k}{2} \rfloor$ for every $j \in [r]$. We start with the case when k is even.

$$\text{Ave } c(Q)^2 = \frac{4}{n^2} \text{Ave} \left(\sum_{x \in Q} x \right)^2.$$

This is again a linear combination of terms x^2 for $x \in P$ and $2xy$ for $x, y \in P$, $x \neq y$ of distinct colors and $2xy$ for $x, y \in P$, $x \neq y$ of the same color. It is clear that each x^2 goes with coefficient $\frac{1}{2}$, each $2xy$ from different colors with coefficient $\frac{1}{4}$ while the coefficient of $2xy$ with x, y of the same color (and $x \neq y$) is

$$\frac{\binom{k/2}{2}}{\binom{k}{2}} = \frac{k-2}{4(k-1)} = \frac{1}{4}\left(1 - \frac{1}{k-1}\right).$$

Thus, writing $\sum_{(1)}$ (resp. $\sum_{(2)}$) for the sum taken over pairs x, y of distinct color (resp. of the same color),

$$\begin{aligned}
\text{Ave} \left(\sum_{x \in Q} x \right)^2 &= \frac{1}{2} \sum_{x \in P} x^2 + \frac{1}{4} \sum\nolimits_{(1)} 2xy + \\
&\qquad \frac{1}{4}\left(1 - \frac{1}{k-1}\right) \sum\nolimits_{(2)} 2xy \\
&= \frac{1}{4} \sum_{x \in P} x^2 + \frac{1}{4} \left(\sum_{x \in P} x \right)^2 - \frac{1}{4(k-1)} \sum\nolimits_{(2)} 2xy \\
&= \frac{1}{4}\left(1 - \frac{1}{k-1}\right) \sum_{x \in P} x^2 - \frac{1}{4(k-1)} \sum_{j=1}^{k} \left(\sum_{x \in C_j} x \right)^2 \\
&\leq \frac{1}{4} \frac{k}{k-1} \sum_{x \in P} x^2 \leq \frac{1}{4} \frac{k}{k-1} \frac{nD^2}{2},
\end{aligned}$$

according to Lemma 4.1. Returning now to $\text{Ave}\, c(Q)^2$ we have

$$\text{Ave}\, c(Q)^2 = \frac{4}{n^2} \text{Ave} \left(\sum_{x \in Q} x \right)^2 \leq \frac{kD^2}{2(k-1)n} = \frac{1}{k-1} \frac{D^2}{2r}$$

Consequently there is a $Q \subset P$ satisfying (i). Assume next that k is odd: $k = 2s + 1$ say, with $s \geq 1$. So the average is to be taken over all subsets Q of P with $|Q \cap C_j| = s$ for every $j \in [k]$, and

$$\text{Ave}\, c(Q)^2 = \frac{1}{(sr)^2} \text{Ave} \left(\sum_{x \in Q} x \right)^2.$$

The coefficients of x^2 and $2xy$ in $\text{Ave} \left(\sum_{x \in Q} x \right)^2$ are determined the same way as before and we have

$$\begin{aligned}
\text{Ave} \left(\sum_{x \in Q} x \right)^2 &= \frac{s}{2s+1} \sum_{x \in P} x^2 + \\
&\qquad \frac{s^2}{(2s+1)^2} \sum\nolimits_{(1)} 2xy + \frac{s(s-1)}{(2s+1)2s} \sum\nolimits_{(2)} 2xy \\
&= \frac{s}{2s+1} \Bigg(\sum_{x \in P} x^2 + \frac{s}{2s+1} \sum\nolimits_{(1)} 2xy + \\
&\qquad \frac{s-1}{2s} \sum\nolimits_{(2)} 2xy \Bigg) \\
&= \frac{s}{2s+1} \Bigg[\frac{s+1}{2s+1} \sum_{x \in P} x^2 + \frac{s}{2s+1} \left(\sum_{x \in P} x \right)^2 + \\
&\qquad \left(\frac{s-1}{2s} - \frac{s}{2s+1} \right) \sum\nolimits_{(2)} 2xy \Bigg] \\
&= \frac{s(s+1)}{(2s+1)^2} \left[\sum_{x \in P} x^2 - \frac{1}{2s} \sum\nolimits_{(2)} 2xy \right] \\
&= \frac{s(s+1)}{(2s+1)^2} \left[\left(1 - \frac{1}{2s}\right) \sum_{x \in P} x^2 - \frac{1}{2s} \sum_{1}^{k} \left(\sum_{x \in C_j} x \right)^2 \right] \\
&\leq \frac{s(s+1)}{(2s+1)^2} \left(1 - \frac{1}{2s}\right) \sum_{x \in P} x^2 \leq \frac{(s+1)(2s-1)}{2(2s+1)^2} \frac{nD^2}{2},
\end{aligned}$$

where Lemma 4.1 is used again. As $n = rk = r(2s+1)$,

$$\begin{aligned}
\text{Ave}\, c(Q)^2 &\leq \frac{1}{s^2 r^2} \frac{(s+1)(2s-1)}{2(2s+1)^2} \frac{nD^2}{2} \\
&= \frac{(k+1)(k-2)}{k(k-1)^2} \frac{D^2}{2r} < \frac{1}{k-1} \frac{D^2}{2r},
\end{aligned}$$

where the last inequality follows easily from $k \geq 3$. This proves part (ii).

Corollary 6.1. *Under the conditions of Lemma 6.1 there is a partition Q_0, Q_1 of P with $|Q_0 \cap C_j| = \lfloor \frac{k}{2} \rfloor$ and $|Q_1 \cap C_j| = \lceil \frac{k}{2} \rceil$ for every $j \in [r]$ such that $|c(Q_0)| = |c(Q_1)|$ when k is even, and $\frac{k-1}{k+1}|c(Q_0)| = |c(Q_1)|$ when k is odd. Moreover*

$$d(c(Q_1), c(P)) \leq d(c(Q_0), c(P)) \leq \frac{1}{\sqrt{k-1}} \frac{D}{\sqrt{2r}}.$$

Proof. Set $Q_0 = Q$ where Q comes from Lemma 6.1 and $Q_1 = P \backslash Q_0$. In the even case $c(Q_0)+c(Q_1) = 2c(P) = o$ again. In the odd case $sc(Q_0)+(s+1)c(Q_1) = nc(P) = o$, implying that $|c(Q_1)| = \frac{k-1}{k+1}|c(Q_0)|$. Moreover, $n = rk$ and in all cases $|c(Q_1)| \leq |c(Q_0)| \leq \frac{1}{\sqrt{k-1}} \frac{D}{\sqrt{2r}}$.

Proof. (Theorem 2.5) We build an incomplete binary tree. Its root is P and its vertices are subsets of P. The children of P are Q_0, Q_1 from the above Corollary, the children of Q_0 resp. Q_1 are Q_{00}, Q_{01} and Q_{10}, Q_{11} obtained again by applying Corollary 6.1 to Q_0 and Q_1. And we split the resulting sets into two parts as equal sizes as possible the same way. And so on. We stop when the set $Q_{\delta_1 \dots \delta_h}$ contains exactly one element from each color class. In the end we have sets $P_1, \dots, P_r$ at the leaves. They form a partition of P with $|P_i \cap C_j| = 1$ for every $i \in [r]$ and $j \in [k]$. We have to estimate $d(c(P_i), c(P))$. Let $P, Q^1, \dots, Q^h, P_i$ be the sets in the tree on the path from the root to P_i. Using the Corollary together with triangle inequality gives

$$\begin{aligned} d(c(P), c(P_i)) &\leq d(c(P), c(Q^1)) + d(c(Q^1), c(Q^2)) + \\ &\quad \dots + d(c(Q^h), c(P_i)) \\ &\leq \left[\frac{1}{\sqrt{k-1}} + \frac{1}{\sqrt{\lfloor k/2 \rfloor - 1}} + \right. \\ &\quad \left. \frac{1}{\sqrt{\lfloor k/4 \rfloor - 1}} + \dots \right] \frac{D}{\sqrt{2r}} \\ &\leq \left(1 + \sqrt{2}\right) \frac{D}{\sqrt{r}} = \gamma \frac{D}{\sqrt{r}}, \end{aligned}$$

as one can check easily.

Remark. We mention that with a little extra care the constant $1+\sqrt{2} = 2.4142..$ can be brought down to 2.02.

7 Proof of Theorem 2.6, Theorem 2.8 and Theorem 2.10

Proof. (Theorem 2.8) The proof goes by induction on k and the case $k = 1$ is trivial. For the induction step $(k-1) \to k$ fix a point $q \in \mathbb{R}^d$ and consider the system $\mathcal{F}_1, \dots, \mathcal{F}_{k-1}$. By the induction hypothesis it has at least $\prod_1^{k-1} m_i$ transversal $\mathcal{S} = \{K_1, \dots, K_{k-1}\}$ with $d(q, K(\mathcal{S})) > \sqrt{\rho_1^2 + \dots + \rho_{k-1}^2}$. If $K(\mathcal{S}) = \emptyset$, then one can extend $\mathcal{S}$ by any $K_k \in \mathcal{F}_k$ to the transversal $\mathcal{T} = \mathcal{S} \cup K_k$ and $d(q, K(\mathcal{T})) = \infty > \rho$. This means that $\mathcal{S}$ with $K(\mathcal{S}) = \emptyset$ can be extended to a suitable $\mathcal{T}$ in n_k different ways.

Suppose now that $K(\mathcal{S}) \neq \emptyset$ and let p be the point in $K(\mathcal{S})$ nearest to q. Note that $K(\mathcal{S})$ is contained in the halfspace

$$H = \left\{ x \in \mathbb{R}^d : (p-q)(x-p) \geq 0 \right\}.$$

By our assumption, there are at least m_k sets $K \in \mathcal{F}_k$ with $d(p, K) > \rho_k$. For all such $K = K_k$ consider the transversal $\mathcal{T} = \mathcal{S} \cup K_k$. Then $d(q, K(\mathcal{T})) = \infty > \rho$ if $K(\mathcal{T}) = \emptyset$. Otherwise let p' be the point in $K(\mathcal{T})$ nearest to q. So $p' \in H$ and then

$$\begin{aligned} (p'-q)^2 &= (p'-p)^2 + (p-q)^2 + 2(p'-p)(p-q) \\ &> \rho_k^2 + (\rho_1^2 + \dots + \rho_{k-1}^2). \end{aligned}$$

Thus $\mathcal{S}$ with $K(\mathcal{S}) \neq \emptyset$ extends to a suitable $\mathcal{T}$ in m_k different ways. In both cases $\mathcal{S}$ can be extended to $\mathcal{T}$ in at least $\min\{m_k, n_k\} = m_k$ ways meaning that there are at least $\prod_1^k m_i$ transversals with $d(q, K(\mathcal{T})) > \rho$.

Proof. (Theorem 2.6) We now have a single family $\mathcal{F}$ with $|\mathcal{F}| = n$. For contradiction, assume that for every $p \in \mathbb{R}^d$, there are at least m sets $K \in \mathcal{F}$, for $m > (1-\beta)n$, with $d(p, K) > \frac{\rho}{\sqrt{k}}$. Call an ordered j-tuple $(K_1, \dots, K_j)$ *good* if $\bigcap_1^j K_i$ is disjoint from $B\left(b, \sqrt{\frac{j}{k}}\rho\right)$. We show, by induction on j, that the number of good j-tuples $(K_1, \dots, K_j)$ of $\mathcal{F}$ is greater than

$$(1-\beta)^j n(n-1) \dots (n-j+1).$$

Note that $n(n-1) \dots (n-j+1)$ is the total number of ordered j-tuples of $\mathcal{F}$. Setting $j = k$, we get that the fraction of ordered k-tuples that are good, i.e., k-tuples whose common intersection is disjoint from $B(b, \rho)$, is greater than

$$(1-\beta)^k = (1-\alpha),$$

a contradiction to our assumption on the number of k-tuples intersecting $B(b, \rho)$.

Similar to the induction step of Theorem 2.8, when considering the good $(j-1)$-tuple $(K_1, \dots, K_{j-1})$ of $\mathcal{F}$ we have to consider two cases.

Case 1: $\bigcap_1^{j-1} K_i = \emptyset$. Then we can add any $K \in \mathcal{F}$ distinct from $K_1, \dots, K_{j-1}$. This is altogether $n-j+1$ good j-tuples of $\mathcal{F}$ extending each previous good $(j-1)$-tuple.

Case 2: $\bigcap_1^{j-1} K_i \neq \emptyset$. Let p be the nearest point in $\bigcap_1^{j-1} K_i$ to b. As we are considering good $(j-1)$-tuples, we have

$$|b-p| > \sqrt{\frac{j-1}{k}}\rho.$$

There are m sets $K \in \mathcal{F}$ with $d(p, K) > \frac{\rho}{\sqrt{k}}$ and such a K is different from every K_i because K_i contains p. Fix one such K, and let p' be the nearest point from b to the intersection of $\bigcap_1^{j-1} K_i$ with K. We have

$$|b-p'| > \sqrt{\frac{j-1}{k}\rho^2 + \frac{\rho^2}{k}} = \sqrt{\frac{j}{k}}\rho.$$

Thus this gives altogether m good j-tuples of $\mathcal{F}$ extending each previous good $(j-1)$-tuple.

Thus each good $(j-1)$-tuple is extended to a good j-tuple in either m or $n-j+1$ ways, finishing the induction.

Proof. (Theorem 2.10) Given P, let $\mathcal{C}$ be the family of all convex objects containing more than $\frac{(k-1)}{k}n$ points of P. Note that all k-tuples of elements of $\mathcal{C}$ must contain a point of P in common—the total number of incidences between any k-tuple $\mathcal{C}'$ of objects of $\mathcal{C}$ and points of P is more than $(k-1)\cdot n$, and thus a point of P must lie in greater than $(k-1)$ objects of $\mathcal{C}'$, in other words, all of them. Furthermore, as each point of P lies in $B(o,1)$, the common intersection of every k-tuple of $\mathcal{C}$ must intersect $B(o,1)$. Thus applying Theorem 2.7, one gets the existence of a point q such that $B\left(q, \frac{1}{\sqrt{k}}\right)$ intersects every element of $\mathcal{C}$. This is the required no-dimension centerpoint.

Consider any closed half-space h^+ containing $B(q, \frac{1}{\sqrt{k}})$. If it contains less than $\frac{n}{k}$ points of P, then its complement open half-space h^- contains greater than $n - \frac{n}{k} = \frac{(k-1)}{k}n$ points, say the set P', of P. But then $d(q, \operatorname{conv} P') > \frac{1}{\sqrt{k}}$, a contradiction to the choice of q.

8 Proofs of Theorem 2.11 and Theorem 2.12

Proof. (Theorem 2.11) This is a combination of the Lemma 4.2 and the no-dimension Tverberg theorem, like in [3]. We assume that $n = kr$ (k an integer) by discarding at most $r-1$ points from P. The no-dimension Tverberg theorem implies that P has a partition $\{P_1, \ldots, P_k\}$ such that $\operatorname{conv} P_i$ intersects the ball $B\left(q, \gamma' \frac{D}{\sqrt{r}}\right)$ for every $i \in [k]$, where $q \in \mathbb{R}^d$ is a suitable point and $\gamma' = 2 + \sqrt{2}$.

Next choose a sequence $1 \le j_1 \le j_2 \le \ldots \le j_r \le k$ (repetitions allowed) and apply Lemma 4.2 to the sets $P_{j_1}, \ldots, P_{j_r}$ where we have to set $\eta = \frac{\gamma'}{\sqrt{r}}$. This gives a transversal $T_{j_1 \ldots j_r}$ of $P_{j_1}, \ldots, P_{j_r}$ whose convex hull intersects the ball

$$B\left(q, \frac{D}{\sqrt{2r}}\sqrt{1 + 2(r-1)\eta^2}\right).$$

The radius of this ball is

$$\frac{D}{\sqrt{2r}}\sqrt{1 + 2\frac{r-1}{r}\gamma'^2} \le \frac{D}{\sqrt{r}}\sqrt{6.5 + 4\sqrt{2}} < 3.5\frac{D}{\sqrt{r}}$$

as the function under the first large square root sign is decreasing with r and for $r = 2$ it is $13 + 8\sqrt{2}$. So the convex hull of all of these transversals intersects $B\left(q, \frac{3.5D}{\sqrt{r}}\right)$. They are all distinct r-element subsets of P and their number is

$$\binom{k+r-1}{r} = \binom{\frac{n-s}{r} + r - 1}{r} \ge r^{-r}\binom{n}{r},$$

as one can check easily.

Proof. (Theorem 2.12) This is an algorithm that goes along the same lines as in the original weak ϵ-net theorem [1]. Set $F := \emptyset$ and let $\mathcal{H}$ be the family of all r-tuples of P. On each iteration we will add a point to F and remove r-tuples from $\mathcal{H}$.

If there is $Y \subset P$ with $\left(F + \frac{3.5D}{\sqrt{r}}B\right) \cap \operatorname{conv} Y = \emptyset$, then apply Theorem 2.11 to that Y resulting in a point $q \in \mathbb{R}^d$ such that the convex hull of at least

$$\frac{1}{r^r}\binom{\epsilon n}{r}$$

r-tuples from Y intersect $B\left(q, \frac{3.5D}{\sqrt{r}}\right)$. Add the point q to F and delete all r-tuples $Q \subset Y$ from $\mathcal{H}$ whose convex hull intersects $B\left(q, \frac{3.5D}{\sqrt{r}}\right)$. On each iteration the size of F increases by one, and at least $r^{-r}\binom{\epsilon n}{r}$ r-tuples are deleted from $\mathcal{H}$. So after

$$\frac{\binom{n}{r}}{\frac{1}{r^r}\binom{\epsilon n}{r}} \le \frac{r^r}{\epsilon^r}$$

iterations the algorithm terminates as there can't be any further $Y \subset P$ of size ϵn with $\left(F + \frac{3.5D}{\sqrt{r}}B\right) \cap \operatorname{conv} Y = \emptyset$. Consequently the size of F is at most $r^r\epsilon^{-r}$.

Acknowledgements. K. Adiprasito was supported by ERC StG 716424 - CASe and ISF Grant 1050/16. I. Bárány was supported by the Hungarian National Research, Development and Innovation Office NKFIH Grants K 111827 and K 116769, and by ERC-AdG 321104. N. Mustafa was supported by the grant ANR SAGA (JCJC-14-CE25-0016-01).

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