On a Problem of Danzer

Nabil H. Mustafa
Université Paris-Est, Laboratoire d’Informatique Gaspard-Monge, Equipe A3SI, ESIEE Paris
mustafan@esiee.fr

Saurabh Ray
Department of Computer Science, NYU Abu Dhabi, United Arab Emirates
saurabh.ray@nyu.edu

Abstract

Let $C$ be a bounded convex object in $\mathbb{R}^d$, and $P$ a set of $n$ points lying outside $C$. Further let $c_p, c_q$ be two integers with $1 \leq c_q \leq c_p \leq n - \left\lfloor \frac{d}{2} \right\rfloor$, such that every $c_p + \left\lfloor \frac{d}{2} \right\rfloor$ points of $P$ contains a subset of size $c_q + \left\lfloor \frac{d}{2} \right\rfloor$ whose convex-hull is disjoint from $C$. Then our main theorem states the existence of a partition of $P$ into a small number of subsets, each of whose convex-hull is disjoint from $C$. Our proof is constructive and implies that such a partition can be computed in polynomial time.

In particular, our general theorem implies polynomial bounds for Hadwiger-Debrunner $(p, q)$ numbers for balls in $\mathbb{R}^d$. For example, it follows from our theorem that when $p > q \geq (1 + \beta) \cdot \frac{d}{\pi}$ for $\beta > 0$, then any set of balls satisfying the HD$(p, q)$ property can be hit by $O\left(q^2 p^{1+\frac{1}{\beta}} \log p\right)$ points. This is the first improvement over a nearly 60-year old exponential bound of roughly $O\left(2^d\right)$.

Our results also complement the results obtained in a recent work of Keller et al. where, apart from improvements to the bound on HD$(p, q)$ for convex sets in $\mathbb{R}^d$ for various ranges of $p$ and $q$, a polynomial bound is obtained for regions with low union complexity in the plane.

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1 Introduction

Given a finite set $C$ of geometric objects in $\mathbb{R}^d$, we say that $C$ satisfies the HD$(p, q)$ property if for any set $C' \subseteq C$ of size $p$, there exists a point in $\mathbb{R}^d$ common to at least $q$ objects of $C'$. The goal then is to show that there exists a small set $Q$ of points in $\mathbb{R}^d$ such that each object of $C$ contains some point of $Q$; such a $Q$ is called a hitting set for $C$.

These bounds for a set $C$ of convex sets in $\mathbb{R}^d$ have been studied since the 1950s (see the surveys [7, 8, 15]), and it was only in 1991 that Alon and Kleitman [1], in a breakthrough

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result, gave an upper-bound that is independent of $|C|$. Unfortunately it depends exponentially on $p, q$ and $d$. For the case where $C$ consists of arbitrary convex objects, the current best bounds remain exponential in $p, q$ and $d$.

**Theorem A** ([1, 9]). Let $C$ be a finite set of convex objects in $\mathbb{R}^d$ satisfying the HD$(p, q)$ property, where $p, q$ are two integers with $p \geq q \geq d + 1$. Then there exists a hitting set for $C$ of size

$$
\begin{cases}
O\left(p^{d^{p+1}} \cdot \log^c d^d \log d \cdot p\right), \\
(p - q) + O\left(\left(\frac{p}{q}\right)^d \log^c d^d \left(\frac{p}{q}\right)\right), \\
p - q + 2,
\end{cases}
$$

for $q \geq \log p$ and $q \geq p^{1/4 + \epsilon}, p \geq p(d, \epsilon)$.

where $c$ is an absolute constant independent of $|C|, p, q$ and $d$, and $p(d, \epsilon)$ is a function depending only on $d$ and $\epsilon$.

Consider the basic case where $C$ is a set of balls in $\mathbb{R}^d$ satisfying the HD$(p, q)$ property. Theorem A implies—ignoring logarithmic factors and for general values of $p$ and $q$—the existence of a hitting set of size no better than $O\left(p^d\right)$. Furthermore, it requires $q \geq d + 1$—a necessary condition for arbitrary convex objects but not for balls.

Almost 60 years ago, Danzer [4, 5] considered the HD$(p, q)$ problem for balls. The best bound that we are aware of, derived from the survey of Eckhoff [7] by combining inequalities (4.2), (4.4) and (4.5), is stated below. It is better than the one from Theorem A quantitatively, but also in that it gives a bound requiring only that $q \geq 2$. Further, for a very specific case—namely when $p = q$ and $(d - q)$ is $O(\log d)$—it succeeds in giving polynomial bounds.

**Theorem B** ([7]). Let $B$ be a finite set of balls in $\mathbb{R}^d$. If $B$ satisfies the HD$(p, q)$ property for some $d \geq p \geq q \geq 2$, then there exists a hitting set for $B$ of size at most

$$
\sqrt{\frac{3\pi}{2}} \cdot 2^{d - q} \cdot \left((p - q) \cdot 2^q \cdot d^2 \cdot g(d) + 4(d - q + 2)^2 \cdot g(d - q + 2)\right)
$$

where $g(x) = \log x + \log \log x + 1$. Ignoring logarithmic terms, the above bound is of the form $\Theta\left((p - q) \cdot 2^d \cdot d^2 + 2^{d - q} \cdot (d - q)^2\right)$. If $p \neq q$ the first term dominates, otherwise the second term dominates.

Turning towards the lower-bound for the case where $C$ is a set of unit balls in $\mathbb{R}^d$, Bourgain and Lindenstrauss [2] proved a lower-bound of $1.0645^d$ when $p = q = 2$ in $\mathbb{R}^d$, i.e., one needs at least $1.0645^d$ points to hit all pairwise intersecting unit balls in $\mathbb{R}^d$.

**Our Result**

We consider a more general set up for the HD$(p, q)$ problem, as follows.

Let $C$ be a convex object in $\mathbb{R}^d$, and $P$ a set of $n$ points lying outside $C$. For each $p \in P$, let $H_p$ be the set of hyperplanes separating $p$ from $C$. Let $C_p$ be the set of points in $\mathbb{R}^d$ dual to the hyperplanes in $H_p$ (see [12, Chapter 5.1]), and let $S = \{C_p : p \in P\}$.

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2 There are easy examples, e.g. when the convex objects are hyperplanes in $\mathbb{R}^d$. 
Our goal is to study the HD(p, q) property for S—namely, that out of every p objects of S, there exists a point in \( \mathbb{R}^d \) common to at least q of them. This is equivalent to the property of C and P that out of every p-sized set \( P' \subseteq P \), there exists a hyperplane separating C from a q-sized subset \( P'' \subseteq P' \)—or equivalently, \( \text{conv}(P'') \) is disjoint from C.

Our main theorem is the following. For a simpler expression, let \( c_p, c_q \) be two positive integers such that \( p = c_p + \left\lfloor \frac{d}{2} \right\rfloor \) and \( q = c_q + \left\lfloor \frac{d}{2} \right\rfloor \).

\[ \lambda_d(c_p, c_q) = K_2 \frac{d}{c_q} \cdot \left( \sqrt{2} K_1 \right)^{\frac{d}{2}} \cdot \left( \lfloor d/2 \rfloor + c_q \right)^2 \cdot \left( \lfloor d/2 \rfloor + c_p \right)^{1 + \frac{\lfloor d/2 \rfloor - 1}{c_q}} \cdot \log(\lfloor d/2 \rfloor + c_p) \]

sets, each of whose convex-hull is disjoint from C. Here \( K_1, K_2 \) are absolute constants independent of n, d, \( c_p \) and \( c_q \). Furthermore, such a partition can be computed in polynomial time.

The proof, presented in Section 2, is a combination of three ingredients: the Alon-Kleitman technique [1], bounds on independent sets in hypergraphs [9] and bounds on \((\leq k)\)-sets for half-spaces [3]. It is an extension of the proof in [14] which studied Carathéodory’s theorem in this setting.

**Remark 1:** The restriction that \( q \geq \left\lfloor \frac{d}{2} \right\rfloor + 1 \) is necessary—as can be seen when \( P \) form the vertices of a cyclic polytope in \( \mathbb{R}^d \) and C is a slightly shrunk copy of \( \text{conv}(P) \).

**Remark 2:** Note that when \( c_q \geq \beta \cdot \frac{d}{2} \), for any absolute constant \( \beta > 0 \), the above bound is polynomial in the dimension d—it is upper-bounded by \( O\left(q^2 p^{1+\frac{d}{2}} \log p \right) \).

**Remark 3:** It was shown in [13] that \( C_p \) is a convex object in \( \mathbb{R}^d \) and thus the bounds of Theorem A apply. As before, Theorem 1 substantially improves upon this, as the bounds following from Theorem A are exponential in d and furthermore, require \( q \geq d + 1 \).

As an immediate corollary of Theorem 1, we obtain the first improvements to the old bound on the \((p, q)\) problem for balls in \( \mathbb{R}^d \). The bound in Theorem B is exponential in d—except in special cases where \( p = q \) and \((d - q)\) is \( O(\log d) \). On the other hand, our result gives polynomial bounds as long as \( q \geq \beta d \) for any constant \( \beta > \frac{1}{2} \).

**Corollary 2 (Hadwiger-Debrunner \((p, q)\) bound for balls in \( \mathbb{R}^d \)).** Let \( B \) be collection of balls in \( \mathbb{R}^d \) such that for every subset of \( c_p + \left\lfloor \frac{d+1}{2} \right\rfloor \) balls in \( B \), some \( c_q + \left\lfloor \frac{d+1}{2} \right\rfloor \) have a common intersection, where \( c_p \) and \( c_q \) are integers such that \( 1 \leq c_q \leq c_p \leq n - \left\lfloor \frac{d+1}{2} \right\rfloor \). Then there exists a set \( X \) of \( \lambda_{d+1}(c_p, c_q) \) points that form a hitting set for the balls in \( B \). Here \( \lambda_{d+1}(\cdot, \cdot) \) is the function defined in the statement of Theorem 1.

**Proof.** Observe that one can stereographically ‘lift’ balls in \( \mathbb{R}^d \) to caps of a sphere \( S \) in \( \mathbb{R}^{d+1} \), where a cap of a sphere is a portion of the sphere contained in a half-space that doesn’t

\[ 3 \text{ Recall that Theorem B assumes } q \leq p \leq d. \]
contain the center of the sphere. Thus we will prove a slightly more general result where $\mathcal{B}$ consists of caps of a $d$-dimensional sphere $S$ embedded in $\mathbb{R}^{d+1}$.

For a point $x \in S$, let $h_x$ denote the hyperplane tangent to $S$ at $x$. For any point $y$ lying outside $S$, define the separating set of $y$ to be

$$S_y = \{ z \in S : h_z \text{ separates } y \text{ and } S \}.$$ 

Geometrically, $S_y$ is the set of points of $S$ ‘visible’ from $y$, and form a cap of $S$. Furthermore, for any cap $K$ of $S$, there is a unique point $w$ such that $K = S_w$. We denote this point $w$ by apex($K$).

Given the set of caps $\mathcal{B}$ on $S$, consider the point set

$$\text{apex}(\mathcal{B}) = \{ \text{apex}(B) : B \in \mathcal{B} \}.$$ 

Observe that for a point $x \in S$ and a cap $B \in \mathcal{B}$, $x \in B$ if and only if $x \in S_{\text{apex}(B)}$. As $\mathcal{B}$ satisfies the $(p, q)$ property—namely that for every $p$-sized subset $B'$ of $\mathcal{B}$, there exists a point $x \in S$ lying in some $q$ elements of $B'$—we have that for every $p$-sized subset $A'$ of $\text{apex}(\mathcal{B})$, there exists a point $x \in S$ lying in the separating set of some $q$ points of $A'$. In other words, $h_x$ separates these $q$ points from $S$.

Applying Theorem 1 with $C = S$ and $P = \text{apex}(\mathcal{B})$ in dimension $d + 1$, we conclude that $P$ can be partitioned into a family $\Xi$ of $\lambda_{d+1}(c_p, c_q)$ sets, each of whose convex hull is disjoint from $S$. Consider a set $P' \in \Xi$. Since the convex hull of $P'$ is disjoint from $S$, we can find a hyperplane $h_x$ tangent to $S$ at $x$ such that $h_x$ separates $P'$ from $S$. This implies that all the caps in $\mathcal{B}$ corresponding to the points in $P'$ contain the point $x$. Thus for each set of $\Xi$ we obtain a point which is contained in all the caps corresponding to the points in that set. These $|X| = \lambda_{d+1}(c_p, c_q)$ points form the required set $X$. \hfill \qed

Our results complement the recent results of Keller, Smorodinsky and Tardos [9, 10] who obtain polynomial bounds for regions of low union complexity in the plane.

## 2 Proof of Theorem 1

Given a set $P$ of points in $\mathbb{R}^d$ and an integer $k \geq 1$, a set $P' \subseteq P$ is called a $k$-set of $P$ if $|P'| = k$ and if there exists a half-space $h$ in $\mathbb{R}^d$ such that $P' = P \cap h$. A set $P' \subseteq P$ is called a $(\leq k)$-set if $P'$ is a $l$-set for some $l \leq k$. The next well-known theorem gives an upper-bound on the number of $(\leq k)$-sets in a point set (see [17]).

\begin{theorem}[Clarkson-Shor [3]] For any integer $k \geq \lfloor \frac{d}{2} \rfloor + 1$, the number of $(\leq k)$-sets of any set of $n$ points in $\mathbb{R}^d$ is at most

$$\kappa_d(n, k) = 2 \left( \frac{K_1}{[d/2]} \right)^{[d/2]} \left( \frac{n}{[d/2]} \right)^{k + [d/2]} \leq \kappa'_d(k) \cdot n^{[d/2]},$$

where $\kappa'_d(k) = 2K_1^{[d/2]} \left( 1 + \frac{k}{[d/2]} \right)^{[d/2]}$ and $K_1 \geq e$ is an absolute constant independent of $n$, $d$ and $k$.

\end{theorem}

\begin{definition}[Depth] Given a set $P$ of $n$ points in $\mathbb{R}^d$ and any set $Q \subseteq P$, define the depth of $Q$ with respect to $P$, denoted $\text{depth}_P(Q)$, to be the minimum number of points of $P$ contained in any half-space containing $Q$.

\end{definition}
For two parameters \( l \geq k \geq 2 \), let \( \tau_d(n,k,l) \) denote the maximum number of subsets of size \( k \) and depth at most \( l \) with respect to \( P \) in any set \( P \) of \( n \) points in \( \mathbb{R}^d \):

\[
\tau_d(n,k,l) = \max_{P \subseteq \mathbb{R}^d} \left| \{Q \subseteq P : |Q| = k \text{ and } \text{depth}_P(Q) \leq l \} \right| .
\]

The following statement is easily implied by an application of the Clarkson-Shor technique [3] (e.g., see [16]).

**Theorem 5.** For parameters \( l \geq k \geq \lceil \frac{d}{2} \rceil + 1, \)

\[
\tau_d(n,k,l) \leq e \cdot \kappa_d(n,k) \cdot t^{k- \lfloor d/2 \rfloor},
\]

where the function \( \kappa(\cdot, \cdot) \) is as defined in Equation (1).

**Proof.** Let \( P \) be any set of \( n \) points in \( \mathbb{R}^d \). Let \( t \) be the number of sets of \( P \) of size \( k \) and depth at most \( l \). Pick each element of \( P \) independently with probability \( \rho = \frac{1}{t} \) to get a random sample \( R \). The expected number of \( k \)-sets in \( R \) satisfies

\[
\rho^k \cdot (1 - \rho)^{1-k} \cdot t \leq E \left[ \text{number of } k \text{-sets in } R \right]
\]

\[
\leq 2 \left( \frac{K_1}{[d/2]} \right)^{\lceil \frac{d}{2} \rceil} E \left[ \left( \binom{R}{\lfloor \frac{d}{2} \rfloor} \right)^k \left( \binom{n}{\lceil \frac{d}{2} \rceil} \right)^{\lfloor \frac{d}{2} \rfloor} \right]
\]

\[
= 2 \left( \frac{K_1}{[d/2]} \right)^{\lceil \frac{d}{2} \rceil} \left( \frac{n}{\lceil \frac{d}{2} \rceil} \right) \cdot \rho^{1-k} \left( \frac{n}{\lceil \frac{d}{2} \rceil} \right)^k \left( \frac{d}{2} \right)^{\lfloor \frac{d}{2} \rfloor}
\]

\[
= \kappa_d(n,k) \cdot \rho^{1-k} \left( \frac{n}{\lceil \frac{d}{2} \rceil} \right)^{\lfloor \frac{d}{2} \rfloor}
\]

\[
\implies t \leq \kappa_d(n,k) \cdot \rho^{1-k} \leq e \cdot \kappa_d(n,k) \cdot t^{k- \lfloor d/2 \rfloor},
\]

as \((1 - \frac{1}{t})^{l-k} \leq e \) for any \( l \geq k \geq 2 \). △

**Lemma 6.** Let \( C \) be a bounded convex object in \( \mathbb{R}^d \), and \( P \) a set of \( n \) points lying outside \( C \). Let \( p \geq q \geq \lceil \frac{d}{2} \rceil + 1 \) be parameters such that for every subset \( Q \subseteq P \) of size \( p \), there exists a set \( Q' \subset Q \) of size \( q \) such that \( Q' \) can be separated from \( C \) by a hyperplane. Then there exists a hyperplane separating at least

\[
(2q \cdot p^{q-1} \cdot e \cdot \kappa_d(q))^{\frac{1}{\lfloor \frac{d}{2} \rfloor - q}}
\]

fraction of the points of \( P \) from \( C \).

**Proof.** From [6, 9], it follows that the number of distinct \( q \)-tuples of \( P \) that can be separated from \( C \) by a hyperplane is, assuming that \( n \geq 2p \), at least

\[
\frac{n - p + 1}{n - q + 1} \cdot \binom{n}{p-q} \geq \frac{n^q}{2q \cdot p^{q-1}}.
\]

Let \( l \) be the maximum depth (Definition 4) of any of these separable \( q \)-tuples. The number of such tuples is therefore at most \( \tau_d(n,q,l) \). Thus by Theorem 5 we must have

\[
\frac{n^q}{2q \cdot p^{q-1}} \leq \tau_d(n,q,l) \leq e \cdot \kappa_d(n,q) \cdot l^{q- \lfloor d/2 \rfloor}.
\]
Re-arranging the terms and from inequality (1), we get

\[
l \geq \left( \frac{n^q}{2q \beta^{q-1} \cdot e \cdot \kappa_d (n, q)} \right)^{1/(d/2 + 1)} \geq \left( \frac{n^q}{2q \beta^{q-1} \cdot e \cdot \kappa'_d (q) \cdot n \frac{1}{(d/2 + 1)}} \right)^{1/(d/2 + 1)} = n \cdot \left( 2q \beta^{q-1} \cdot e \cdot \kappa'_d (q) \right)^{1/(d/2 + 1)}.
\]

Thus one of the separable \( q \)-tuples, say \( P' \subseteq P \), must have depth at least \( l \); in other words, the hyperplane separating \( P' \) from \( C \) must contain at least \( l \) points of \( P \). This is the required hyperplane.

We now prove a weighted version of the above statement.

**Corollary 7.** Let \( C \) be a bounded convex object in \( \mathbb{R}^d \), and \( P \) a weighted set of \( n \) points lying outside \( C \), where the weight of each point \( p \in P \) is a non-negative rational number. Let \( p \geq q \geq \left[ \frac{d}{2} \right] + 1 \) be parameters such that for every subset \( Q \subseteq P \) of size \( p \), there exists a set \( Q' \subset Q \) of size \( q \) such that \( Q' \) can be separated from \( C \) by a hyperplane. Then there exists a hyperplane separating a set of points whose weight is at least

\[
\alpha_d(p,q) = \left( 2e \cdot \kappa'_d (q) \cdot q^q \cdot \beta^{q-1} \right)^{1/(d/2 + 1)}
\]

fraction of the total weight of the points in \( P \).

**Proof.** By appropriately scaling all the rational weights, we may assume that each weight is a non-negative integer and we replace a point with weight \( m \) by \( m \) unweighted copies of the point. Let \( P' \) be the new set of points. Observe that any set \( S \) of \( pq \) points in \( P' \) either contains \( q \) copies of some point in \( P \) or it contains \( p \) distinct points from \( P \). In either case, there is hyperplane separating \( q \) points of \( S \) from \( C \). Thus, we can apply Lemma 6 to the point set \( P' \) with the parameter \( p \) in the lemma replaced by \( pq \). The result follows.

Finally we return to the proof of the main theorem.

**Theorem 1.** Let \( C \) be a bounded convex object in \( \mathbb{R}^d \) and \( P \) a set of \( n \) points lying outside \( C \). Further let \( c_p, c_q \) be two integers, with \( 1 \leq c_p \leq c_q \leq n - \left[ \frac{d}{2} \right] \), such that for every \( c_p + \left[ \frac{d}{2} \right] \) points of \( P \), there exists a subset of size \( c_q + \left[ \frac{d}{2} \right] \) whose convex-hull is disjoint from \( C \). Then the points of \( P \) can be partitioned into

\[
\lambda_d(c_p, c_q) = K_2 \frac{d}{c_q} \cdot \left( \sqrt{2}K_1 \right)^{\frac{d}{c_q}} \cdot ([d/2] + c_q)^2 \cdot ([d/2] + c_p) \cdot \log ([d/2] + c_p)
\]

sets, each of whose convex-hull is disjoint from \( C \). Here \( K_1, K_2 \) are absolute constants independent of \( n, d, c_p \) and \( c_q \). Furthermore, such a partition can be computed in polynomial time.

**Proof.** Let \( p = c_p + [d/2] \) and \( q = c_q + [d/2] \). Let \( \mathcal{H} \) be the set of all hyperplanes separating a distinct subset of points of \( P \) from \( C \). As the number of subsets of \( P \) is finite, one can assume that \( \mathcal{H} \) is also finite. Consider the following linear program on \( |\mathcal{H}| \) variables \( \{u_h \geq 0 : h \in \mathcal{H}\} \):

\[
\min \sum_{h \in \mathcal{H}} u_h, \quad \text{such that} \quad \forall r \in P: \sum_{h \in \mathcal{H}} u_h \geq 1, \quad (2)
\]

\( h \) separates \( r \) from \( C \).
The LP-dual to the above program, on \(|P|\) variables \(\{w_r \geq 0: r \in P\}\), is:

\[
\max \sum_{p \in P} w_p, \quad \text{such that} \quad \forall h \in \mathcal{H}: \sum_{r \in P} w_r \leq 1.
\]

(3)

Consider an optimal solution \(w^*_p\) of the dual linear program and interpret \(w^*_p\) as the weight of each \(r \in P\). Since the weights are rational, by Corollary 7, there exists a hyperplane \(h \in \mathcal{H}\) separating a subset of \(P\) of combined weight at least \(\epsilon = \alpha_d(p,q)\) fraction of the total weight of all the points. Since the total weight of the points in any half-space is constrained to be at most 1 by the linear program, the total weight of all the points of \(P\) must be at most \(\frac{1}{\epsilon}\). In other words, the optimal value of linear program (3) is at most \(\frac{1}{\epsilon}\). Since the optimal values of both linear programs are equal due to strong duality, the optimal value of linear program (2) is also at most \(\frac{1}{\epsilon}\).

Let \(u^*\) be the optimal solution of linear program (2). If we interpret \(u_h\) as the weight of the hyperplane \(h\), the constraints of the program imply that each point is separated by a set of hyperplanes in \(\mathcal{H}\) whose combined weight is at least 1 out of a total weight of at most \(\frac{1}{\epsilon}\)-in other words, at least \(\epsilon\)-th fraction of the total weight of \(\mathcal{H}\). By associating with each hyperplane the half-space bounded by it and not containing \(C\), and using the \(\epsilon\)-net theorem for half-spaces in \(\mathbb{R}^d\) (see [11]), there exists a set of \(O\left(\frac{d}{\epsilon} \log \frac{d}{\epsilon}\right)\) hyperplanes which together separate all points of \(P\) from \(C\). Recalling that

\[
\frac{1}{\epsilon} = \frac{1}{\alpha_d(p,q)} = \left(2e \kappa'_d(q) q^d p^{q-1}\right) = \left(2e \kappa'_d(q) q^d p^{q-1}\right)^{\frac{1}{q^2}},
\]

and that \(\kappa'_d(q) = 2K_1^d [d/2]^{-[d/2]} \left(1 + \frac{q}{[d/2]}\right)^{[d/2]}\), we get

\[
\frac{1}{\epsilon} = \left(4K_1^d [d/2]^{-[d/2]} \left(1 + \frac{q}{[d/2]}\right)^{[d/2]} q^d p^{q-1}\right)^{\frac{1}{q^2}}
\]

\[
\leq \left(4K_1^{d+1} [d/2]^{-d} (c_q + d)^{[d/2]} q^d p^{q-1}\right)^{\frac{1}{q^2}} \quad \text{(using } e \leq K_1 \text{ and } q = c_q + \lfloor d/2 \rfloor)\]

\[
\leq \left(4K_1^{d+1} [d/2]^{-d} (c_q + d)^{[d/2]} q^d p^{q-1}\right)^{\frac{1}{q^2}}
\]

\[
= O \left(K_1^{d+1} [d/2]^{-d} (c_q + d)^{[d/2]} (c_q + \lfloor d/2 \rfloor)^{1 + \frac{[d/2]}{c_q}} (c_q + \lfloor d/2 \rfloor)^{1 + \frac{[d/2]-1}{c_q}}\right)
\]

\[
= O \left(K_1^{d+1} [d/2]^{-d} (c_q + d)^{[d/2]} (c_q + \lfloor d/2 \rfloor)^{1 + \frac{[d/2]}{c_q}} (c_q + \lfloor d/2 \rfloor)^{1 + \frac{[d/2]-1}{c_q}}\right)
\]

\[
= O \left(K_1^{d+1} [d/2]^{-d} (c_q + d)^{[d/2]} (c_q + \lfloor d/2 \rfloor)^{1 + \frac{[d/2]}{c_q}} (c_q + \lfloor d/2 \rfloor)^{1 + \frac{[d/2]-1}{c_q}}\right)
\]

\[
= O \left(\sqrt{2}K_1^{d+1} [d/2]^{-d} (c_q + d)^{[d/2]} (c_q + \lfloor d/2 \rfloor)^{1 + \frac{[d/2]-1}{c_q}}\right).
\]
The Big-Oh notation here does not hide dependencies on \( d \)—namely we do not treat \( d \) as a constant. From the above it follows that
\[
\log \frac{1}{\epsilon} = O \left( c_q^{-1} \left( \lfloor d/2 \rfloor + c_q \right) \log \left( \lfloor d/2 \rfloor + c_p \right) \right).
\]
Thus, \( \frac{2}{\epsilon} \log \frac{1}{\epsilon} \) is
\[
O \left( \frac{d}{c_q} \left( \sqrt{2} \frac{K_1}{\epsilon} \right)^{\frac{d}{c_q}} \left( \lfloor d/2 \rfloor + c_q \right) \left( \lfloor d/2 \rfloor + c_p \right)^{1+\frac{\lfloor d/2 \rfloor - 1}{c_q}} \log \left( \lfloor d/2 \rfloor + c_p \right) \right) 
\]
\[
= O \left( \frac{d}{c_q} \left( \sqrt{2} K_1 \right)^{\frac{d}{c_q}} \left( \lfloor d/2 \rfloor + c_q \right)^2 \left( \lfloor d/2 \rfloor + c_p \right)^{1+\frac{\lfloor d/2 \rfloor - 1}{c_q}} \log \left( \lfloor d/2 \rfloor + c_p \right) \right).
\]
Since linear programs can be solved in polynomial time and epsilon nets can be computed in polynomial time, the partition of \( P \) into the above number of sets can be achieved in polynomial time. The theorem follows. □

References

