1.1 Helly’s Theorem and its Applications

One of the fundamental theorems on convexity is Helly’s Theorem, which states the following:

**Theorem 1** (Helly’s Theorem). Given a set \( C \) of compact convex objects in \( \mathbb{R}^d \) such that every \((d+1)\) of them have a common intersection, all of them have a common intersection.

**Intuitive sketch.** Consider the proof for the one-dimensional case, where \( C \) becomes a set of intervals. Our proof will be by induction on \( n = |C| \), the number of intervals. Let \( C = \{C_1, \ldots, C_n\} \) be the set of pair-wise intersecting intervals. Let \( I = \cap_{i=1}^{n-1} C_i \) be the common intersection interval of the first \( n-1 \) intervals of \( C \). By inductive hypothesis, \( I \neq \emptyset \). Now we need to show that the remaining interval \( C_n \) intersects \( I \). Otherwise, say \( C_n \) lies to the right of \( I \) (the case where it lies to the left of \( I \) is similar). Note the following structural fact: the right endpoint of \( I \) is also the right endpoint of an interval \( C_i \in C \setminus C_n \). Then this \( C_i \) and \( C_n \) do not have a common intersection, a contradiction.

The generalization to \( \mathbb{R}^2 \) is immediate by a similar structural claim: given a set of convex polygons \( C \), let \( I \) be their common intersection. See Figure 1.1 (a). Then any fixed vertex, say \( v \), of \( I \) is the intersection of two edges, say \( e_1, e_2 \), of the corresponding two objects of \( C \), say polygons \( C_1 \) and \( C_2 \). Furthermore, the halfspace \( h_i \) supporting the edge \( e_i \), for \( i = 1, 2 \), and containing \( I \) also contains \( C_i \) completely.

Proceed via induction, as before, on the cardinality of \( C \). Let \( I = \cap_{i=1}^{n-1} C_i \). Suppose that \( C_n \) does not intersect \( I \). Then, there exists a plane \( h \) separating \( I \) from \( C_n \). See Figure 1.1 (b). By translating \( h \) towards \( I \), we can assume it passes through some vertex, say vertex \( v \), of \( I \). The common intersection of the two convex objects whose boundary edges define \( v \) lies within the intersection of their corresponding halfspaces, and on the same side of \( h \) as \( I \). Therefore, the common intersection of these two objects with \( C_n \) is empty, a contradiction to the fact that every 3-tuple must have a common intersection.

![Figure 1.1](image.png)

Figure 1.1: a) Each halfspace \( h_i \) completely contains \( C_i \), b) The line \( h \) separating \( C_n \) from \( I \) contains \( h_1 \cap h_2 \).
We now give a formal proof by an extremal configuration argument. We first present the simplest case – pair-wise intersecting intervals in $\mathbb{R}$ – and then generalize it to the $d$-dimensional case.

**Extremal configuration argument.** Let $C$ be a set of intervals in $\mathbb{R}$. Pick the interval, say $C' \in C$, whose left endpoint is as to the right as possible. Formally, define $h(C)$ to be the coordinate of the left endpoint of interval $C$, and let $h(C) = \max_{C \in C} h(C)$. Let $C'$ be the interval of $C$ which realizes $h(C)$. We claim that the point $h(C')$ is contained in all intervals. Assuming otherwise, an interval $C''$ not containing $h(C')$ either lies to the left of the point $h(C')$, in which case $C'$ does not intersect $C''$, or it lies to the right, in which case $C'$ does not realize the maximum of $h(C)$. Either way, we get a contradiction and we’re done.

Now let $C$ be a set of convex objects in $\mathbb{R}^d$. Define $h(C')$, for any $C' \subseteq C$, to be the $y$-coordinate of the point with the lowest $y$-coordinate in the common intersection of $C'$. Define

$$h(C) = \max_{|C'| = d} h(C')$$

Let $C'$ be the set of $d$ objects of $C$ which realize $h(C)$, and let $p$ be the point in their common intersection with the lowest $y$-coordinate (i.e., $p$ has $y$-coordinate equal to $h(C)$). We claim that $p$ is contained in all convex objects. Otherwise, assume some object $C$ does not contain $p$. Since every $(d+1)$-tuple intersects, $C' \cup C$ has non-empty intersection. Furthermore, note that every point of this common intersection has $y$-coordinate greater than that of $p$: the smallest $y$-coordinate of the common intersection of $C'$ is that of $p$ and $C$ does not contain $p$. Now it is fairly intuitive (see Figure 1.2(a) for an illustration in $\mathbb{R}^2$) that from this $(d+1)$-sized intersection of $C' \cup C$, whose lowest point has $y$-coordinate greater than that of $p$, one can pick $d$ objects whose lowest $y$-coordinate of the intersection is greater than that of $p$, a contradiction to the definition of $p$.

Figure 1.2: a) Any convex object $C$ (dotted) not containing $p$ but intersecting $C_1 \cap C_2$ must have $h(\{C, C_1\}) > h(\{C_1, C_2\})$ or $h(\{C, C_2\}) > h(\{C_1, C_2\})$, b) $C'_{d-1}$ for two-dimensional case, where any two disjoint intervals must form a pair with greater $h(\cdot)$. 

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Here’s the formal argument: consider convex objects obtained by the intersection of each object of \( \mathcal{C}' \cup \mathcal{C} \) with the plane \( h \) defined by \( y = h(\mathcal{C}) \), i.e., all points with \( y \)-coordinate equal to that of \( p \). Call this set of convex objects in \( \mathbb{R}^{d-1} \) as \( \mathcal{C}_{d-1} \). As the common intersection of \( \mathcal{C}' \cup \mathcal{C} \) lies above this plane, the common intersection of \( \mathcal{C}_{d-1} \) in \( h \) is empty, and so by (the contra-positive statement of) Helly’s theorem in \( \mathbb{R}^{d-1} \), there exist \( d \) objects of \( \mathcal{C}_{d-1} \), say \( \mathcal{C}'' \), whose common intersection lies strictly above \( h \). But then we have \( h(\mathcal{C}'') > h(\mathcal{C}') \), a contradiction. See Figure 1.2(b). 

There are several equivalent ways of stating Helly’s theorem, here’s an important one:

**Theorem 2 (Locality of Convexity).** Given a set \( \mathcal{C} \) of convex objects in \( \mathbb{R}^d \) having a non-empty common intersection \( I \), consider the point \( p \) of \( I \) with the minimum \( y \)-coordinate. Then \( p \) is also the point with the minimum \( y \)-coordinate in the common intersection of some \( d \) objects of \( \mathcal{C} \).

The proof of this statement was essentially given in our proof of Helly’s theorem.

**Helly’s Theorem \( \implies \) Locality of Convexity:** Given \( \mathcal{C} \), pick the \( d \)-tuple \( \mathcal{C}' \) with the maximum \( h(\mathcal{C}') \). Let \( p \) be the point realizing \( h(\mathcal{C}') \). We complete our proof if we show that every object of \( \mathcal{C} \) contains \( p \). Let \( h \) be the plane defined by \( y = h(\mathcal{C}') \). Assume an object \( C \) does not contain \( p \). Since the intersection of \( \mathcal{C}' \) lies on and above \( h \), the common intersection of \( \mathcal{C}' \cup C \) lies strictly above \( h \), and so the common intersection of the set \( \mathcal{C}_{d-1} = \{ C' \cap h, \ C' \in \mathcal{C}' \cup \mathcal{C} \} \) is empty. By Helly’s theorem in \( \mathbb{R}^{d-1} \), there exists a \( d \)-sized subset of \( \mathcal{C}_{d-1} \) which also has empty intersection, i.e., a \( d \)-sized subset of \( \mathcal{C}' \cup C \) has common intersection strictly above \( h \), a contradiction to the maximality of \( \mathcal{C}' \).

**Locality of Convexity \( \implies \) Helly’s Theorem:** Given \( \mathcal{C} \) such that every \( (d+1) \)-tuple have a common intersection, pick the \( d \)-tuple \( \mathcal{C}' \) with maximum \( h(\mathcal{C}') \). Let \( p \) be this point in the intersection of \( \mathcal{C}' \) realizing \( h(\mathcal{C}') \). We claim that \( p \) is contained in every convex object. Assume a \( C \) does not contain \( p \). The common intersection of \( \mathcal{C}' \cup C \) lies above \( h \), and by the Locality of Convexity, there exists a \( d \)-sized subset of \( \mathcal{C}' \cup C \) with the common intersection above \( h \), a contradiction to \( h(\mathcal{C}') \).

**Remark:** We defined \( h(\cdot) \) as maximizing the minimum \( y \)-coordinate. All the proofs work when over any direction, instead of just the vertical direction. Similarly, Locality of Convexity holds for the extreme vertex in any direction.

**The centerpoint theorem.** Helly’s theorem is the starting point of a large number of basic results. Here’s one that we’ll encounter several times again.

**Theorem 3 (Centerpoint Theorem).** Given any set \( P \) of \( n \) points in \( \mathbb{R}^d \), there exists a point \( c \in \mathbb{R}^d \) such that any closed halfspace containing \( c \) contains at least \( \frac{n}{d+1} \) points of \( P \).

The connection to Helly’s theorem comes from the following statement. Given \( P \), let \( \mathcal{C} \) denote the set of all convex polytopes containing greater than \( \frac{dn}{d+1} \) points of \( P \). We can assume \( \mathcal{C} \)
to be finite by requiring each polytope to be the convex hull of the points it contains. Now, note the following: any point that hits all the convex polytopes in $C$ is a centerpoint, and vice versa. Let $c$ be a centerpoint, and assume a polytope $C$ is not hit by $c$. Then there exists a plane separating $c$ from $C$, and the closed halfspace defined by this plane containing $c$ contains less than $\frac{n}{d+1}$ points of $P$, a contradiction. The other direction is similar.

Now we only need to show that $C$ can be hit by one point. To use Helly’s theorem, one needs to show that every $(d + 1)$-tuple of $C$ has a common point of $P$. This follows from a counting argument. For contradiction, assume a $(d + 1)$-tuple $C' \subseteq C$ has an empty common intersection. Now count all pairs of type $(p, C)$, where $p \in P$, $C \in C'$ and $p \in C$. This is greater than $|C'| \cdot \frac{dn}{d+1}$, and at most $d \cdot n$. Putting them together gives a contradiction.

The centerpoint theorem now follows from Helly’s theorem.

**Remark:** Even though every $(d+1)$-tuple of $C$ has a point of $P$ in common, Helly’s theorem does not then imply that a point of $P$ has to be a centerpoint. In fact, that is false; e.g., take $P$ to be in convex position.

**Remark:** The centerpoint theorem is optimal: take $n$ points, and put them very near the vertices of a regular simplex in $\mathbb{R}^d$, in groups of size $\frac{n}{d+1}$. It is not hard to see that no point can do better than $\frac{n}{d+1}$. Of course, for some pointsets, one can get a better bound. The depth of a pointset is defined as the largest number $\frac{1}{d+1} \leq q \leq 1/2$ such that there exists a point $c$ such that any closed halfspace containing $c$ contains at least $qn$ points of $P$. So for example, the depth of a set of points spread uniformly around a circle is $1/2$ (realized by its center). The point that realizes the depth of $P$ is sometimes called the Tukey point.

**Final Remarks.** The proof of Locality of Convexity given in [M] uses Helly’s theorem in $\mathbb{R}^d$ instead of in $\mathbb{R}^{d-1}$, which is not really required. The proof of Helly’s theorem in [M] is derived from Radon’s theorem, and though while clever, is somewhat non-intuitive. It uses induction on $n$ to construct the point hitting all convex objects, and is computationally infeasible. A good side-product of the given proof is that computing the point hitting all convex objects is trivial: just enumerate over all $d$-sized subsets of $C$, and select $h(C)$. Finally, we will see two more proofs of Helly’s theorem during the course (in different contexts): using a smallest-ball argument, and using Brouwer’s fixed point theorem.