1.3 Radon’s, Caratheodory’s and Colorful Caratheodory’s Theorems

**Theorem 6** (Radon’s Theorem). Given a set $P$ of $(d + 2)$ points in $\mathbb{R}^d$, one can always partition $P$ into sets $P_1$ and $P_2$ such that $\text{conv}(P_1) \cap \text{conv}(P_2) \neq \emptyset$.

The case in $\mathbb{R}$ is easy: given three points on the real line, say $p_1, p_2, p_3$ ordered from left to right, choose the two sets as $\{p_2\}, \{p_1, p_3\}$. The proof in $\mathbb{R}^2$ is also simple: construct the convex hull of $P$. Either it has all four points on the convex hull, i.e., its a quadrilateral, and so pick the diagonal points. Otherwise, the points on the convex hull form one set, and the points inside form the second one.

We now present the general proof by induction on $d$. Given the set $P$ of $(d + 2)$ points in $\mathbb{R}^d$, if $\text{conv}(P)$ contains a point, say $p'$, of $P$, we are done by picking the two partitions to be $\{p'\}$ and $P \setminus \{p'\}$. Similarly, if there exists a plane containing $d + 1$ points of $P$, we are again done by applying Radon’s theorem for these points in $\mathbb{R}^{d-1}$.

Otherwise, find a plane $h$ spanned by $d$ points $P'$ that has the remaining two points on different sides of $h$. Then the line segment between the two points in $P \setminus P'$ intersects $h$, say in point $q$. Now consider the $d + 1$ points $P' \cup q$, all lying on $h$. By Radon’s theorem in $\mathbb{R}^{d-1}$, one can partition $P' \cup q$ into two sets, say $P'_1$ and $P'_2$, whose convex hull intersect, say at the point $r$. Assume $q \in P'_1$. Now replace $q$ by the two points in $P \setminus P'$. Since $q$ is in the convex hull of these two points, by replacing $q$ with $P \setminus P'$, one can only increase the convex hull of $P'_1$. So the convex hulls of $P'_1 \setminus q \cup \{P \setminus P'\}$ and $P'_2$ have non-empty intersection. See Figure 1.3.

It remains to show that it is always possible to find the required plane $h$: take the convex hull of $P$, and pick any point on it, say $p_1 \in P$. Now the convex hull of $P \setminus p_1$ does not contain $p_1$, and so there exists the required plane defined by a facet of $\text{conv}(P \setminus p_1)$ separating $p_1$ from $\text{conv}(P \setminus p_1)$.

![Figure 1.3](image1.png)

Figure 1.3: The plane $h$ defined by three points $p_1, p_2, p_3$, with the points $p_4$ and $p_5$ on different sides of $h$. The final partition is $\{p_2, p_4, p_5\}$ and $\{p_1, p_3\}$. 

10
Another basic theorem is the following.

**Theorem 7** (Caratheodory’s Theorem). Given a point set $P$ in $\mathbb{R}^d$, and a point $q \in \text{conv}(P)$, there exists a $(d + 1)$-point subset $P'$ such that $q \in \text{conv}(P')$.

The statement is easy in $\mathbb{R}^2$: pick any point from the convex hull of $P$, and draw line segments from it to every other vertex of the convex hull. This partitions the convex hull into triangles. Just pick the triangle that contains $q$. See Figure 1.4 (a).

The proof in $\mathbb{R}^d$ is similar: pick any point $p_1$ on the convex hull of $P$, and imagine shooting a half-infinite ray from $p_1$ to $q$. The line segment $p_1q$ lies inside the convex hull. The ray continues after passing $q$ and eventually will hit a facet of the convex hull in point $q'$. By Caratheodory’s theorem in $\mathbb{R}^{d-1}$, $q'$ is contained in the convex hull of $d$ points, say $P''$, lying on this facet. Then the $(d + 1)$-sized set $P'' \cup p_1$ contains $q$. See Figure 1.4 (b).

**Theorem 8** (Colorful Caratheodory’s Theorem). Let $P_1, \ldots, P_{d+1}$ be $d + 1$ sets in $\mathbb{R}^d$, each containing $(d + 1)$ points, such that $q \in \text{conv}(P_i)$ for all $i$. Then, there exists a $(d + 1)$-sized subset $P'$ such that $q \in \text{conv}(P')$, where $P'$ contains one point from each $P_i$.

Any simplex defined by $d + 1$ points, one from each $P_i$, is called a *colorful* simplex. The proof is by an extremal configuration argument. Pick the colorful simplex, say $S$, which is the closest to $q$. If $S$ contains $q$, we are done. Otherwise let $p$ be the closest point in $S$ to $q$. Let $h$ be the half-plane orthogonal to the direction $\vec{qp}$ at the point $p$. Clearly, $S$ lies in one closed halfspace defined by $h$, and $q$ in the other. Note that a few, or all, points of $S$ may lie on $h$. But by Caratheodory’s theorem in $\mathbb{R}^{d-1}$, there exists a at-most $d$-sized subset on $h$ that also contains $p$. Pick any one of the remaining points of $S$, say point $r$ belonging to the set $P_i$. Now since $\text{conv}(P_i)$ contains $q$, at least one point of $P_i$, say $r'$, must lie on the same side of $h$ as $q$. But this yields a contradiction, since the colorful simplex defined by removing $r$ and adding $r'$ has a smaller distance to $q$ than $S$.  

11
Final remarks. The proof of Radon’s in [M] is actually shorter than the one we give, and follows directly from the definition of affine dependence: imagine that $P$ can be partitioned into $P_1$ and $P_2$, both of whose convex hulls contain a common point, say $q$. Then $q$ can be written as a convex combination of points of $P_1$, and also as a separate convex combination of points of $P_2$. Putting these two convex combinations as equal, one gets the exact definition of affine dependence. To prove Radon’s, simply reverse the above steps.

The proof of Colorful Caratheodory’s is exactly the same as in [M], and written very nicely, so you should read it there.