Lecture 4: Using Brouwer’s fixed point theorem

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A measure of data depth

Given a set $L$ of $n$ lines in the plane, does there always exist a ‘deep’ point?
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Regression depth

Given a set $L$ of $n$ planes in $\mathbb{R}^d$, define $RD(q)$ for $q \in \mathbb{R}^d$ as the minimum number of planes any half-infinite ray from $q$ must intersect.
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RD(L) = \max_{q \in \mathbb{R}^d} RD(q)
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**Question:** Can one always find a point of high regression depth?
The regression-depth theorem

Regression-depth theorem

For any set $L$ of $n$ planes in $\mathbb{R}^d$, one can find a point of regression-depth at least $\lceil n/(d + 1) \rceil$. And there are examples where one cannot do any better.
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Proof of regression-depth theorem

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Proof of regression-depth theorem

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We now look at an even simpler proof of this theorem by Karasev (2008).

It uses a topological theorem called Brouwer’s fixed point theorem, which we first describe.
Brouwer’s fixed point theorem

Given any continuous function $f: B^n \rightarrow B^n$, there exists a point $p \in B^n$ such that $f(p) = p$. This is called a fixed point of $f$. 
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Brouwer’s fixed point theorem in $\mathbb{R}$

Any continuous function $f : [0, 1] \rightarrow [0, 1]$ has a fixed point.
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Back to the centerpoint theorem

To see its beauty and power, let's re-prove the centerpoint theorem.
Centerpoint theorem

Given a set $P$ of $n$ points in $\mathbb{R}^d$, there exists a point $q \in \mathbb{R}^d$ such that any halfspace containing $q$ contains at least $n/(d + 1)$ points of $P$. 
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- First construct all convex polytopes containing $\geq dn/(d + 1)$ points
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- Any point lying in all such polytopes is the required point
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- Every $(d + 1)$-tuple of the polytopes have a common intersection
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- Every $(d + 1)$-tuple of the polytopes have a common intersection

Only have to prove that given a set of convex polytopes where every $(d + 1)$-tuple has a non-empty intersection, all of them have a non-empty intersection. Known as Helly's theorem.
Proof of Helly’s theorem

Let the set of polytopes be \( C = \{ C_1, \ldots, C_m \} \)
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Let the set of polytopes be $\mathcal{C} = \{ C_1, \ldots, C_m \}$
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For a point $p$, let $q_1, \ldots, q_m$ be the $m$ closest points to each of the polytopes. $f(p)$ maps $p$ to the centroid of $\{q_1, \ldots, q_m\}$. 
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By Brouwer’s fixed point theorem, $f$ has a fixed point, say $p$. 
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By Brouwer’s fixed point theorem, \( f \) has a fixed point, say \( p \).

Claim: \( p \) lies in all polytopes of \( \mathcal{C} \)
Proof of Helly’s theorem

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Claim: \( p \) lies in all polytopes of \( C \)

Centroid of the \( m \) closest points in \( C \) to \( q \) is \( q \) itself
Proof of Helly’s theorem

By Brouwer’s fixed point theorem, $f$ has a fixed point, say $p$.

Claim: $p$ lies in all polytopes of $\mathcal{C}$

A centroid lies in the convex-hull of its points
Proof of Helly’s theorem

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Claim: $p$ lies in all polytopes of $\mathcal{C}$

Therefore $q$ lies in the convex-hull of some $(d + 1)$ closest points
Proof of Helly’s theorem

By Brouwer’s fixed point theorem, $f$ has a fixed point, say $p$.

**Claim:** $p$ lies in all polytopes of $C$

The $(d + 1)$ polytopes have empty common intersection, a contradiction.
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A similar problem

\( \mathcal{C} = \{ C_1, \ldots, C_n \} \) are \( n \) disjoint convex polygons in \( \mathbb{R}^2 \).
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Does there always exist a ‘shallow’ point?
Intersecting Rays Theorem

Given a set $\mathcal{C}$ of $n$ disjoint convex polytopes in $\mathbb{R}^d$, there exists a point $q \in \mathbb{R}^d$ such that any half-infinite ray from $q$ intersects at most $dn/(d + 1)$ objects of $\mathcal{C}$. 

... and one really can't do much better.
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Intersecting rays for unit balls

Intersecting Rays Theorem

Given a set $C$ of $n$ disjoint unit balls in $\mathbb{R}^d$, there exists a point $q \in \mathbb{R}^d$ such that any half-infinite ray from $q$ intersects at most $dn/(d + 1)$ objects of $C$. 

Proof?
Intersecting rays for unit balls

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**Claim:** $f$ is continuous
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We now apply Brouwer’s fixed point theorem to $f$ ... except one has to verify two things:

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Claim: $f$ is continuous

- The closest point function $f_C(p) = \arg\min_{q \in C} d(p, q)$ is continuous
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But what happens after we take a centerpoint of \( \{q_1, \ldots, q_m\} \)?
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- For any set $P$, there can be many centerpoints
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But what happens after we take a centerpoint of $\{q_1, \ldots, q_m\}$?
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- So if we pick an arbitrary centerpoint, $f$ need not be continuous
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But what happens after we take a centerpoint of $\{q_1, \ldots, q_m\}$?

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- Need to have a ‘uniform’ way of choosing a centerpoint
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**Claim:** The set of centerpoints form a convex polytope
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- Need to have a ‘uniform’ way of choosing a centerpoint

Claim: The set of centerpoints form a convex polytope

Take the mean of the centerpoint region for the points $\{q_1, \ldots, q_m\}$. 
Proof of Intersecting Rays Theorem

By Brouwer’s fixed point theorem, \( f \) has a fixed point, say \( p \).
Proof of Intersecting Rays Theorem

By Brouwer’s fixed point theorem, $f$ has a fixed point, say $p$.

**Claim:** Any half-infinite ray from $p$ intersects at most $dn/(d + 1)$ objects.
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By Brouwer’s fixed point theorem, \( f \) has a fixed point, say \( p \).

**Claim:** Any half-infinite ray from \( p \) intersects at most \( dn/(d + 1) \) objects.
Proof of Intersecting Rays Theorem

By Brouwer’s fixed point theorem, $f$ has a fixed point, say $p$.

Claim: Any half-infinite ray from $p$ intersects at most $dn/(d + 1)$ objects.

Have to prove that any ray $\vec{r}$ intersects at most $dn/(d + 1)$ objects.
Proof of Intersecting Rays Theorem

By Brouwer’s fixed point theorem, $f$ has a fixed point, say $p$.

Claim: Any half-infinite ray from $p$ intersects at most $dn/(d + 1)$ objects

Consider the plane $h$ orthogonal to $\vec{r}$
Proof of Intersecting Rays Theorem

By Brouwer’s fixed point theorem, \( f \) has a fixed point, say \( p \).

**Claim:** Any half-infinite ray from \( p \) intersects at most \( \dfrac{dn}{d+1} \) objects.

The halfspace \( h^- \) has at most \( \dfrac{n}{d+1} \) closest points on one side.
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Claim: Any half-infinite ray from \( p \) intersects at most \( \frac{dn}{d + 1} \) objects.

Each corresponding object cannot intersect \( \vec{r} \).
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So $\vec{r}$ intersects at most $n - n/(d + 1) = dn/(d + 1)$ objects.
Proof of Regression Depth Theorem

We come back to the proof of the Regression-depth theorem.
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Exercise problem
Proof of Regression Depth Theorem

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**Exercise problem**

**Hint:** Very similar to an earlier proof, except one twist.
QUESTIONS?
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