LATTICE OPERATORS UNDERLYING DYNAMIC SYSTEMS

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Abstract

This paper investigates algebraic and continuity properties of increasing set operators underlying dynamic systems. We recall algebraic properties of increasing operators on complete lattices. The topologies used for the study of continuity properties are defined by lim sup and lim inf and ↑- and ↓-continuity of lattice operators. We apply these notions to several operators induced by differential equation or differential inclusion. We focus especially on the operators Viab_F and Inv_F which with any closed subset K associated its viability kernel $\mathrm{Viab}_F(K)$ and its invariance kernel Inv_F, for a differential inclusion. We provide its algebraic properties, a structure of the set of its fixpoints and continuity properties. At the end, we show that morphological operators used in image processing are particular cases of operators induced by constant differential inclusion.

Key-words: *Complete lattice, algebraic dilation and erosion, algebraic opening and closing, ↑- and ↓-continuity, differential inclusion, contingent cone, reachable set, exit tube, viability kernel, invariance kernel.*

1 Introduction

A complete lattice $(\mathcal{L}, \leq)$ is a partial ordered set such that every subset $\mathcal{H}$ of $\mathcal{L}$ has a supremum and and infimum denoted by $\bigvee \mathcal{H}$ and $\bigwedge \mathcal{H}$. As important examples for the following, we mention:

- The power space $\mathcal{P}(X)$, the set of all subsets of X supplied with the inclusion order $\subset$, $\mathcal{F}(X)$ the space of all closed subsets of X and $\mathcal{C}(X)$ the space of all closed convex subsets of X. The supremum and the infimum on $\mathcal{P}(X)$ and $\mathcal{C}(X)$ are given by:

$$\begin{cases} \forall \mathcal{H} \subset \mathcal{P}(X), & \bigvee \mathcal{H} = \bigcup_{K \in \mathcal{H}} K \quad \textit{and} \quad \bigwedge \mathcal{H} = \bigcap_{K \in \mathcal{H}} K \\ \forall \mathcal{H} \subset \mathcal{F}(X), & \bigvee \mathcal{H} = \overline{\bigcup_{K \in \mathcal{H}} K} \quad \textit{and} \quad \bigwedge \mathcal{H} = \bigcap_{K \in \mathcal{H}} K \\ \forall \mathcal{H} \subset \mathcal{C}(X), & \bigvee \mathcal{H} = \overline{co}\left(\bigcup_{K \in \mathcal{H}} K\right) \quad \textit{and} \quad \bigwedge \mathcal{H} = \bigcap_{K \in \mathcal{H}} K \end{cases}$$

where $\overline{co}(K))$ represents the closed convex hull of K.

- Every function $f : X \to \overline{\mathbb{R}}$ can be characterized by its hypograph $\mathcal{H}p(f)$, or by its epigraph $\mathcal{E}p(f)$. These two sets are defined by : $\mathcal{H}p(f) = \{(x,t) \in X \times \mathbb{R} \mid f(x) \geq t\}$ and $\mathcal{E}p(f) = \{(x,t) \in X \times \mathbb{R} \mid f(x) \leq t\}$. Then the space $\Phi^{usc}(X)$ (resp. $\Phi^{lsc}(X)$) of all upper (resp. lower) semicontinuous functions defined on X with extended values is a complete lattice for the order relationship $\forall f, g \in \Phi^{usc}(X),\ f \leq g \Leftrightarrow \mathcal{H}p(f) \subset \mathcal{H}p(g)$ (resp. $\forall f, g \in \Phi^{lsc}(X),\ f \leq g \Leftrightarrow \mathcal{E}p(f) \subset \mathcal{E}p(g)$), and the supremum and the infimum are defined by:

$$\forall f_i \in \Phi^{usc}(X), \quad \begin{cases} f = \vee f_i & \Leftrightarrow \quad \mathcal{H}p(f) = \overline{\cup \mathcal{H}p(f_i)} \\ f = \wedge f_i & \Leftrightarrow \quad \mathcal{H}p(f) = \cap \mathcal{H}p(f_i) \end{cases}$$

and,

$$\forall f_i \in \Phi^{lsc}(X), \quad \begin{cases} f = \vee f_i & \Leftrightarrow \quad \mathcal{E}p(f) = \overline{\cup \mathcal{E}p(f_i)} \\ f = \wedge f_i & \Leftrightarrow \quad \mathcal{E}p(f) = \cap \mathcal{E}p(f_i) \end{cases}$$

The first part of the present paper, is devoted to algebraic properties of increasing operators (*i.e.* $\psi : \mathcal{L} \mapsto \mathcal{L}$ such that $x \leq y$ implies $\psi(x) \leq \psi(y)$ where $\mathcal{L}$ is a complete lattice for $\leq$ order. If we consider an operator $\psi : \mathcal{L} \mapsto \mathcal{L}$, we can classify it as a dilation, erosion, opening or closing. The main example of these characterization is given by morphological operators in the lattice of closed subsets supplied with the inclusion order. The morphological dilation of a closed subset K by a structuring element B is given by : $K \oplus \check{B} = \{k - b \mid k \in K,\ b \in B\}$. Furthermore, the study of continuity properties of these operators on lattice can be done with the lim sup and lim inf topologies.

In this paper, we focus on set operators defined on a complete lattice (in particular on the power space $\mathcal{P}(X)$, or on the space $\mathcal{F}(X)$ of closed subsets of X), induced by a differential inclusion. This paper links algebraic and continuity properties to operators underlying dynamic systems.
A differential inclusion $x'(t) \in F(x(t))$ is a generalisation of the notion of differential equation $x'(t) = f(x(t))$ where the dynamic is multivalued and non deterministic. The main example is given by control system : $F(x) = \{f(x,u)\}_{u \in U(x)}$. Many set operators can be deduced from the framework of differential inclusions. The operator Viab_F which with any closed subset K associated its "viability kernel"[1] $\mathrm{Viab}_F(K)$ for the differential inclusion $x'(t) \in F(x(t))$, is *increasing antiextensive and idempotent.* We say that it is an *algebraic opening.* We show that it is upper-semi-continuous and we give a structure of the set of its fixpoints. The operator Inv_F which with any closed subset K associated its "invariance

[1] If K is a viability domain, then for any initial state $x_0 \in K$, there exists a solution $x(.)$ on $[0, \infty[$ to differential inclusion $x'(t) \in F(x(t))$ which satisfies $x(t) \in K$. The largest closed subset of K viable under F (which may be empty) is the viability kernel of K for F and we denote it by $\mathrm{Viab}_F(K)$.

kernel"[2] $\text{Inv}_F(K)$ for the differential inclusion $x'(t) \in F(x(t))$, is an algebraic opening which commutes with the infimum. We show that it is upper-semi-continuous and we give its dual operator in sense of the complementation.
In the last part, we will study a particular example: the differential inclusion $x'(t) \in B$ where B is a symmetrical compact convex set. As particular case of the previous results, we provide algebraic and continuity properties of the induced operators which are morphological operators.

2 Lattice framework

In this section, we briefly recall some basic notions and some results that we shall use later on. For more details on lattice theory consult [10, 11, 5].

2.1 Algebraic definitions and properties on a complete lattice

By an operator, we shall mean a mapping of a complete lattice $\mathcal{L}$ into a complete lattice $\mathcal{M}$.

Definition 2.1 *We say that an operator $\psi : \mathcal{L} \mapsto \mathcal{M}$ is:*
- increasing *if $A \leq B$ implies $\psi(A) \leq \psi(B)$,*
- extensive *(resp.* antiextensive*) if $\mathcal{L} = \mathcal{M}$ and $\psi(A) \geq A$ (resp. $\psi(A) \leq A$),*
- idempotent *if $\mathcal{L} = \mathcal{M}$ and $\psi^2 = \psi$.*

It is obvious that if ψ is an increasing operator then for any family $K_i \in \mathcal{L}$, we have $\psi(\bigvee K_i) \geq \bigvee \psi(K_i)$ and $\psi(\bigwedge K_i) \leq \bigwedge \psi(K_i)$.

If $\mathcal{L}$ and $\mathcal{M}$ are subsets of Boolean lattices[3] (for example $\mathcal{F}(X)$ in $\mathcal{P}(X)$) then every element K in $\mathcal{L}$ (resp. in $\mathcal{M}$) has a unique complement in the Boolean lattice which we denote by K^c (in $\mathcal{P}(X)$, we have $K^c = X \setminus K$). The dual operator of an operator $\psi : \mathcal{L} \mapsto \mathcal{M}$ is given by:

$$\psi^* : \mathcal{L}^* \mapsto \mathcal{M}^* \quad \text{where} \quad \psi^*(K) = (\psi(K^c))^c \text{ and } \mathcal{L}^* = \{K^c \mid K \in \mathcal{L}\}$$

It is clear that ψ^* is increasing if and only if ψ is, ψ^* is idempotent if and only if ψ is, and ψ^* is extensive if and only if ψ is anti-extensive.
For example, the space $\mathcal{F}(X)$ of all closed subsets of X and the space $\mathcal{F}^*(X)$ of all open subsets of X are duals, and we have:

$$\begin{cases} \forall \mathcal{H} \subset \mathcal{F}(X), & \bigvee \mathcal{H} = \overline{\bigcup_{K \in \mathcal{H}} K} \quad \textit{and} \quad \bigwedge \mathcal{H} = \bigcap_{K \in \mathcal{H}} K \\ \forall \mathcal{H} \subset \mathcal{F}^*(X), & \bigvee \mathcal{H} = \bigcup_{K \in \mathcal{H}} K \quad \textit{and} \quad \bigwedge \mathcal{H} = Int\left(\bigcap_{K \in \mathcal{H}} K\right) \end{cases}$$

where $Int(K)$ denotes the interior of K.

[2]If K is an invariance domain, then for any initial state $x_0 \in K$, all solutions $x(.)$ on $[0, \infty[$ of the differential inclusion $x'(t) \in F(x(t))$ starting from x_0 which satisfies $x(t) \in K$. The largest closed subset of K invariant under F (which may be empty) is the invariant kernel of K for F and we denote it by $\text{Inv}_F(K)$.

[3]In a complete lattice $\mathcal{L}$, there exists a smallest element denoted by $\emptyset$ and a greatest element denoted by E. If $x, y \in \mathcal{L}$ are such that $x \wedge y = \emptyset$ and $x \vee y = E$, the y is called a complement of x. A lattice $\mathcal{L}$ is called complemented if all elements in $\mathcal{L}$ have a complement. A Boolean lattice is complemented distributive lattice, *i.e.* every element has a unique complement and such that the supremun distributes over the infimum and conversely the infimum distributes over the supremum

2.1.1 Algebraic dilation and erosion

Definition 2.2 [13][8, Definition 2.1.] *We will say that ψ is an* algebraic dilation *(resp. an* algebraic erosion*) if ψ distributes over the suprema (resp. over the infima), i.e. $\psi(\bigvee K_i) = \bigvee \psi(K_i)$ (resp. $\psi(\bigwedge K_i) = \bigwedge \psi(K_i)$).*

In set-valued analysis [4], there are two ways to extend the concept of the inverse image by a set-valued map F of a subset K: $F^{-1}(K) = \{y \in X | \ F(y) \cap K \neq \emptyset\}$ and $F^{+1}(K) = \{y \in X | \ F(y) \subset K\}$. The subset $F^{-1}(K)$ is called the inverse image of K by F and $F^{+1}(K)$ is called the core of K by F. They naturally coincide when F is a single valued, and we observe that $F^{+1}(K^c) = X \setminus F^{-1}(K)$ and $F^{-1}(K^c) = X \setminus F^{+1}(K)$.

Proposition 2.3 *Let $F : X \rightsquigarrow X$ be a set-valued map. The operator $K \mapsto F^{-1}(K)$ is an algebraic dilation and the map $K \mapsto F^{+1}(K)$ is an algebraic erosion on $\mathcal{P}(X)$ and $F^{-1}(.)$ and $F^{+1}(.)$ are dual operators, i.e. $(F^{-1})^* = F^{+1}$.*

Proof: $F^{+1}(\cap K_i) = \{y \in X | \ F(y) \subset (\cap K_i)\} = \{y \in X | \ \forall i, \ F(y) \subset K_i\} = \cap F^{+1}(K_i)$. By complementation, we see that $F^{-1}(\cup K_i) = \cup F^{-1}(K_i)$. □

We give an other example. Let $F : X \rightsquigarrow X$ be a set-valued map, then the operator δ_F (resp. ε_F) defined on $\mathcal{P}(X)$ by:

$$\delta_F(K) = F(K) := \bigcup_{x \in K} F(x) \quad \left(\text{resp. } \varepsilon_F(K) = \bigcap_{x \in K^c} [F(x)]^c\right)$$

is an algebraic dilation (resp. an algebraic erosion) and we have $\delta_F^* = \varepsilon_F$. If $F(x) = B_x$ where B is a subset of X and $B_x = B + x$ represents the translation set of B by the vector x, then $\delta_F(K) = K \oplus B$ and $\varepsilon_F(K) = K \ominus \check{B}$ where $\oplus$ and $\ominus$ are respectively the Minkowski addition and the Minkowski subtraction and $\check{B} = -B$ is the symmetric set of B.

Definition 2.4 [8, Definition 2.2.] *Let $\delta : \mathcal{L} \mapsto \mathcal{M}$ and $\varepsilon : \mathcal{M} \mapsto \mathcal{L}$) be two operators on a complete lattice $\mathcal{L}$. Then we will say that (ε, δ) is an* adjunction *if for every $A \in \mathcal{L}$ and for every $B \in \mathcal{M}$, we have:*

$$\delta(A) \leq B \Leftrightarrow A \leq \varepsilon(B) \tag{1}$$

We will denote $\varepsilon^{\circledast} = \delta$ and $\delta^{\circledast} = \varepsilon$ when (ε, δ) is an adjunction.

If (ε, δ) is an adjunction, it follows automatically that δ is an algebraic dilation and ε an algebraic erosion. For example, in the case of Minkowski operations on $\mathcal{P}(\mathbb{R}^n)$, then the subtraction and the addition by a set B form an adjunction.
An automorphism of $\mathcal{L}$ is both an algebraic dilation and an algebraic erosion and (ψ^{-1}, ψ) and (ψ, ψ^{-1}) are adjunctions.

Proposition 2.5 [8, Theorem 2.7.]

1. *For any algebraic dilation $\delta : \mathcal{L} \mapsto \mathcal{M}$, there exists a unique algebraic erosion $\delta^{\circledast} : \mathcal{M} \mapsto \mathcal{L}$ such that $(\delta^{\circledast}, \delta)$ is an adjunction.*

2. *For any algebraic erosion $\varepsilon : \mathcal{M} \mapsto \mathcal{L}$, there exists a unique algebraic dilation $\varepsilon^{\circledast} : \mathcal{L} \mapsto \mathcal{M}$ such that $(\varepsilon, \varepsilon^{\circledast})$ is an adjunction.*

Let $F : X \rightsquigarrow X$ be a set-valued map. The operator $K \mapsto F(K) = \cup_{x \in K} F(x)$ is a dilation and its adjoint erosion is $F^{\circledast} : K \mapsto F^{+1}(K)$ on $\mathcal{P}(X)$.

Proof: $F(K) \subset H \Leftrightarrow \forall x \in K,\ F(x) \subset H \Leftrightarrow \forall x \in K,\ x \in F^{+1}(H) \Leftrightarrow K \subset F^{+1}(H)$. □

We provide two examples of adjunctions:

- Let $f, g : X \mapsto \overline{\mathbb{R}}$ be two extended functions defined on X with real values. We defined [16] the supconvolution (resp. the infconvolution) of f by g at scale $t \geq 0$ by:

$$f \Cap tg(x) = \sup_{y \in \mathbb{R}^n} \{f(x-ty)+tg(y)\} \quad (\text{resp. } f \uplus tg(x) = \inf_{y \in \mathbb{R}^n} \{f(x+ty)-tg(y)\})$$

  It is obvious that $U_{\Cap} : f \mapsto f \Cap tg$ is an algebraic dilation and $U_{\uplus} : f \mapsto f \uplus tg$ is an algebraic erosion on $\Phi^{usc}(X)$ and $(U_{\uplus}, U_{\Cap})$ is an adjunction.

- Let us consider two set-valued maps $F, G : X \rightsquigarrow X$ on a vector space X and $h > 0$. We say that the map $F \boxplus hG : X \rightsquigarrow Y$ defined by [17, Definition 2.1.]

$$(F \boxplus hG)(x) := \bigcup_{y \in X} (F(x - hy) \oplus hG(y))$$

  is the set-convolution of F and G. The map $F \boxminus hG : X \rightsquigarrow Y$ defined by

$$(F \boxminus hG)(x) := \bigcap_{y \in X} (F(x + hy) \ominus hG(y))$$

  is called the internal set-convolution of F and G.
  It is obvious that the map $\Psi : F \mapsto F \boxplus G$ is an algebraic dilation and $\Phi : F \mapsto F \boxminus G$ is an algebraic erosion for every set-valued map F and G.

Proposition 2.1: *Let $G : X \rightsquigarrow Y$ be a set-valued map. If $\Psi : F \to F \boxplus G$ and $\Phi : F \to F \boxminus G$ then (Φ, Ψ) is an adjunction.*

Proof:

$$\begin{aligned}
\Psi(F) \leq H &\Leftrightarrow \forall x \in X,\ F \boxplus G(x) \subset H(x) \\
&\Leftrightarrow \forall x \in X,\ \cup_y F(x-y) \oplus G(y) \subset H(x) \\
&\Leftrightarrow \forall x, y \in X,\ F(x-y) \oplus G(y) \subset H(x) \\
&\Leftrightarrow \forall x, y \in X,\ F(x-y) \subset H(x) \ominus G(y) \\
&\Leftrightarrow \forall z, x \in X,\ F(z) \subset H(y+z) \ominus G(y) \\
&\Leftrightarrow \forall z \in X,\ F(z) \subset \cap_y H(y+z) \ominus G(y) \\
&\Leftrightarrow F \leq \Phi(H)
\end{aligned}$$

□

2.1.2 Algebraic opening and closing

Definition 2.6 [20, Definition 2.1.] *We will say that $\psi : \mathcal{L} \mapsto \mathcal{L}$ is an* algebraic opening *(resp. an* algebraic closing*) if ψ is increasing, idempotent and antiextensive (resp. extensive).*

The topological opening $K \mapsto Int(K)$ is an algebraic opening on $\mathcal{P}(X)$. The closed convex closure operator $K \mapsto \overline{co}(K)$ is an algebraic closing on $\mathcal{F}(X)$.

Proposition 2.7 [20, Proposition 2.8.] *Given an adjunction (ε, δ), then we have $\delta\varepsilon\delta = \delta$ and $\varepsilon\delta\varepsilon = \varepsilon$. The operator $\delta\varepsilon$ is an algebraic opening and the operator $\varepsilon\delta$ is an algebraic closing.*

Proof:

- Since ε (resp. δ) is an algebraic erosion (resp. dilation), then $\delta\varepsilon$ and $\varepsilon\delta$ are two increasing operators.
- $\varepsilon(x) \leq \varepsilon(x) \Leftrightarrow \delta\varepsilon(x) \leq x$. We deduce that $\delta\varepsilon$ is an anti-extensive operator. With the same statement, we deduce that $\varepsilon\delta$ is an extensive operator.
- Since $\delta\varepsilon(x) \leq x \Leftrightarrow \delta\varepsilon\delta\varepsilon(x) \leq \delta\varepsilon(x)$, but $\varepsilon\delta(x) \geq x \Leftrightarrow \delta\varepsilon\delta\varepsilon(x) \geq \delta\varepsilon(x)$. This yieldsthat $\delta\varepsilon\delta\varepsilon = \delta\varepsilon$. With the same statement, we deduce that $\varepsilon\delta$ is an idempotent operator.

□

Remark 2.8 *We observe that if ψ is an algebraic dilation (resp. an algebraic erosion) then its dual operator ψ^* is an algebraic erosion (resp. an algebraic dilation). Furthermore, if ψ is an algebraic opening (resp. an algebraic closing) then ψ^* is an algebraic closing (resp. an algebraic opening).*

2.1.3 Subset of fixpoints of algebraic opening and closing

Definition 2.9 (Subset of fixpoints) *Let ψ be an operator on $\mathcal{L}$ and $K \in \mathcal{L}$. We say that K is a* fixpoint *of ψ if $\psi(K) = K$. The set of all fixpoints of ψ is called the* subset of fixpoints of ψ *and it is denoted by:*

$$\mathrm{Fix}(\psi) := \{K \in \mathcal{L} \mid \psi(K) = K\}$$

It is obvious that $\mathrm{Fix}(\psi)$ is ψ-closed, because $\forall K \in \mathrm{Fix}(\psi)$ we have $\psi(K) = K$, and $\psi(\psi(K)) = \psi(K)$ implies that $\psi(K) \in \mathrm{Fix}(\psi)$. Then, for every opening $\psi : \mathcal{L} \mapsto \mathcal{L}$, there is an associated subset of fix points. Since ψ is idempotent, $\mathrm{Fix}(\psi)$ is nothing but the image of $\mathcal{L}$ under ψ, *i.e.* $\mathrm{Fix}(\psi) = \psi(\mathcal{L})$.

Lemma 2.10 (Tarski fixpoint theorem, weak version) [7, Theorem 3.3.] *The set* $\mathrm{Fix}(\psi)$ *of fixpoints of an increasing operator ψ on the complete lattice $\mathcal{L}$ is nonempty.*

Theorem 2.11 (Tarski fixpoint theorem) [14, page 122] *Let ψ be an increasing idempotent operator on $\mathcal{L}$. Then* $\mathrm{Fix}(\psi)$ *is a complete lattice included in $\mathcal{L}$.*

Algebraic opening and closing are completely characterized by their subset of fixpoints.

Proposition 2.12 [12, Proposition 7.1.1] *If ψ is an opening, then its subset of fixpoints is closed under suprema, that is if $K_i \in \mathrm{Fix}(\psi)$ for $i \in I$ then $\bigvee_{i\in I} K_i \in \mathrm{Fix}(\psi)$. Conversely, every subset $\mathcal{B}$ of $\mathcal{L}$ which is closed under suprema is the subset of fixpoints of a unique opening ψ given by:*

$$\psi(K) = \bigvee\{B \in \mathcal{B} \mid B \subset K\}$$

For example, in $\mathcal{P}(\mathbb{R}^n)$, the subset of fixpoints of the topological opening $K \mapsto Int(K)$ is the family of all open sets which is closed under union and invariant under translation. Moreover, the interior of a set K is the union of all open balls inside K, *i.e.*

$$Int(K) = \cup\{B \in \mathcal{B} \mid B \subset K\} \quad \text{where } \mathcal{B} \text{ is the family of all open balls}$$

Proposition 2.13 [12, Proposition 7.1.1] *If ϕ is a closing, then its subset of fixpoints is closed under infima, that is if $K_i \in \mathrm{Fix}(\psi)$ for $i \in I$ then $\bigwedge_{i\in I} K_i \in \mathrm{Fix}(\psi)$. Conversely, every subset $\mathcal{B}$ of $\mathcal{L}$ which is closed under infima is the subset of fixpoints of unique closing ϕ given by:*

$$\phi(K) = \bigwedge\{B \in \mathcal{B} \mid B \supset K\}$$

For example, in $\mathcal{P}(\mathbb{R}^n)$, the subset of fixpoints of the closed convex closure $K \mapsto \overline{co}(K)$ is the family of all closed convex sets which is closed under intersection and invariant under translation. Moreover, the closed convex hull of a set K is the intersection of all closed half hyper-planes which contain K, *i.e.*

$$\overline{co}(K) = \cap\{B \in \mathcal{B} \mid K \subset B\} \quad \text{where } \mathcal{B} \text{ is the family of all closed half hyper-planes}$$

Proposition 2.14 [20, Proposition 2.3.] *Let ψ be an algebraic opening and θ be an increasing antiextensive operator. Then the following four statements are equivalent:*

1. $\psi \leq \theta$, *(i.e.* $\forall K \in \mathcal{L}$, $\psi(K) \leq \theta(K)$*),*
2. $\psi\theta = \psi$,
3. $\theta\psi = \psi$,
4. $\mathrm{Fix}(\psi) \subset \mathrm{Fix}(\theta)$.

From this proposition, it follows that an algebraic opening is uniquely determined by its subset of fixpoints.

Corollary 2.15 [20, Corollary 2.4.] *Let ψ_1 and ψ_2 be two algebraic openings, then $\psi_1 = \psi_2$ if and only if $\mathrm{Fix}(\psi_1) = \mathrm{Fix}(\psi_2)$.*

2.2 Order continuity of lattice operators

Throughout this section, we assume that $\mathcal{L}$ is a complete lattice.

Definition 2.16 [9, Definition 2.1.] *For a sequence K_n in $\mathcal{L}$ we define:*

$$\begin{aligned} \liminf K_n &= \bigvee_{N\geq 1} \bigwedge_{n\geq N} K_n \\ \limsup K_n &= \bigwedge_{N\geq 1} \bigvee_{n\geq N} K_n \end{aligned}$$

Obviously, for any sequence K_n, we have $\liminf K_n \leq \limsup K_n$. We say that $K_n \to K$ if $\liminf K_n = \limsup K_n = K$.
It is clear that

$$
\begin{aligned}
\limsup(K_n \bigwedge L_n) &\leq (\limsup K_n) \bigwedge (\limsup L_n) \\
\limsup(K_n \bigvee L_n) &\geq (\limsup K_n) \bigvee (\limsup L_n)
\end{aligned}
$$

and the same relations hold for the lim inf.

For example, on the lattice $\mathcal{F}(X)$ of the closed subsets of X, the lim sup and lim inf are given by:

$$
\begin{aligned}
\limsup K_n &:= \bigcap_{N \geq 1} \left(\overline{\bigcup_{n \geq N} K_n} \right) \\
\liminf K_n &:= \overline{\bigcup_{N \geq 1} \left(\bigcap_{n \geq N} K_n \right)}
\end{aligned}
$$

If $\mathcal{L}$ is a Boolean lattice, we have:

$$
\begin{aligned}
\limsup K_n^c &= (\liminf K_n)^c \\
\liminf K_n^c &= (\limsup K_n)^c
\end{aligned}
$$

Definition 2.17 [9, Definition 3.1.] *Let $\mathcal{L}$ and $\mathcal{M}$ be two complete lattices, and let $\psi : \mathcal{L} \mapsto \mathcal{M}$ be an arbitrary operator. We say that ψ is $\downarrow$-*continuous *if $K_n \to K$ implies that* $\limsup \psi(K_n) \leq \psi(K)$ *and that ψ is $\uparrow$-*continuous *if $K_n \to K$ implies that $\psi(K) \leq$* $\liminf \psi(K_n)$. *If ψ is both $\uparrow$- and $\downarrow$- continuous then $K_n \to K$ implies $\psi(K_n) \to \psi(K)$, and we say that ψ is* continuous.

We can prove that:

Proposition 2.18 [9, Corollary 3.5.]

- *Any algebraic erosion is $\downarrow$-continuous,*
- *Any algebraic dilation is $\uparrow$-continuous.*

In fact this proposition is a corollary of the following proposition:

Proposition 2.19 [9, Proposition 3.4.] *Let ψ be an increasing operator on $\mathcal{L}$. Then the operator ψ is $\downarrow$-continuous if and only if* $\limsup \psi(K_n) \leq \psi(\limsup K_n)$ *for any sequence K_n in $\mathcal{L}$.*

If we now consider the complete lattice $\mathcal{F}(X)$, where X is a topological space which is Hausdorff, locally compact and admits a countable base. On $\mathcal{F}(X)$, we can first define limits of sets introduced by Painlevé in 1902, and called Kuratowski upper and lower limits of sequences of sets:

Definition 2.20 [4, Definition 1.1.1.] *Let $(K_n)_{n\in\mathbf{N}}$ be a sequence of subsets of a metric space X. We say that the subset*

$$\operatorname*{Limsup}_{n\to\infty} K_n := \left\{x \in X \mid \liminf_{n\to\infty} d(x, K_n) = 0\right\}$$

is the upper limit *of the sequence K_n and that the subset*

$$\operatorname*{Liminf}_{n\to\infty} K_n := \{x \in X \mid \lim_{n\to\infty} d(x, K_n) = 0\}$$

is its lower limit. *A subset K is said to be the* limit *or the* set limit *of the sequence K_n if*

$$K = \operatorname*{Liminf}_{n\to\infty} K_n = \operatorname*{Limsup}_{n\to\infty} K_n =: \operatorname*{Lim}_{n\to\infty} K_n$$

Lower and upper limits are obviously closed. We also see at once that $\mathrm{Liminf}_{n\to\infty} K_n \subset \mathrm{Limsup}_{n\to\infty} K_n$ and that the upper limits and lower limits of the subsets K_n and of their closures $\overline{K_n}$ do coincide, since $d(x, K_n) = d(x, \overline{K_n})$, and we have:

$$\begin{aligned}
\operatorname*{Liminf}_{n\to\infty} K_n &= \bigcap_{\varepsilon>0} \bigcup_{N\leq 1} \bigcap_{n\geq N} (K_n \oplus \varepsilon B) \\
\operatorname*{Liminf}_{n\to\infty} K_n &= \bigcap_{\varepsilon>0} \bigcap_{N\leq 1} \bigcup_{n\geq N} (K_n \oplus \varepsilon B)
\end{aligned}$$

where B is the unit ball of X.

Any decreasing sequence of subsets K_n has a limit, which is the intersection of their closures:

$$\text{if } K_n \subset K_m \text{ when } n \geq m, \text{ then } \operatorname*{Lim}_{n\to\infty} K_n = \bigcap_{n\geq 0} \overline{K_n}$$

We have obviously on $\mathcal{F}(X)$, $\mathrm{Limsup}_{n\to\infty} K_n = \limsup_{n\to\infty} K_n$.
It is easy to check that:

Proposition 2.21 [4, Proposition 1.1.2] *If $(K_n)_{n\in\mathbf{N}}$ is a sequence of subsets of a metric space, then* $\mathrm{Liminf}_{n\to\infty} K_n$ *is the* set of limits of sequences $x_n \in K_n$ *and* $\mathrm{Limsup}_{n\to\infty} K_n$ *is the set* of cluster points of sequences $x_n \in K_n$, *i.e., of limits of subsequences $x_{n'} \in K_{n'}$.*

Definition 2.22 [4, Definition 1.4.1.] *Let $\psi : \mathcal{F}(X) \mapsto \mathcal{F}(X)$ be an operator. We say that ψ is* upper-semi-continuous *(u.s.c.) if $K_n \to K$ implies that $\mathrm{Limsup}_{n\to\infty}\psi(K_n) \subseteq \psi(K)$ and that ψ is* lower-semi-continuous *(l.s.c.) if $K_n \to K$ implies that $\psi(K) \subseteq \mathrm{Liminf}_{n\to\infty}\psi(K_n)$. If ψ is both upper and lower semi-continuous then $K_n \to K$ implies $\psi(K_n) \to \psi(K)$, and we say that ψ is* continuous.

Proposition 3.3 of [9] is:

Proposition 2.23 *Let $\psi : \mathcal{F}(X) \mapsto \mathcal{F}(X)$ be an arbitrary operator on $\mathcal{F}(X)$.*

- *If ψ is u.s.c. then ψ is $\downarrow$-continuous,*
- *If ψ is an increasing and $\downarrow$-continuous then ψ is u.s.c.*

We deduce that every increasing erosion is u.s.c. on $\mathcal{F}(X)$.

3 Algebraic properties and differential inclusion

3.1 Differential inclusion and reachable set

In this section, we recall some operators induced by differential inclusions as the reachable map, the exit tube, the viability kernel map and the invariance kernel map. For more details on the differential inclusion theory and viability theory, see [3], [1] or [6]. We give the algebraic properties of these operators.

Control systems are often governed by a family differential equation $x'(t) = f(x(t), u(t))$ where $u(t) \in U(x(t))$. The single-valued map f describes the dynamics of the system: It associates with state x of the system and the control u the velocity $f(x, u)$ of the system. The set-valued map U describes a feedback map assigning to the state x the subset $U(x)$ of admissible controls. If we put $F(x) := f(x, U(x)) = \{f(x,u)\}_{u \in U(x)}$, then the control system is governed by the differential inclusion $x'(t) \in F(x(t))$.

Let us describe the (non deterministic) dynamics of a system by a set-valued map F from the state space X to itself. We consider initial value problems (or Cauchy problems) associated to differential inclusion

$$\text{for almost all } t \in [0,T], x'(t) \in F(x(t)) \tag{2}$$

satisfying the initial condition $x(0) = x_0$.

Let $F : X \rightsquigarrow X$ be a set-valued map from the vector space X to itself. We denote by $\vartheta_F(x)$ the set of solutions $x(.)$ to the differential inclusion :

$$\forall t \in I, \quad x'(t) \in F(x(t)), \qquad x(0) = x \tag{3}$$

starting at the initial state x. We also denote by $\vartheta_F(h, x)$ the set of the values $x(h)$ at time h of the solutions x of (3). For all subsets $K \subset X$, $\vartheta_F(h, K) = \cup_{x \in K} \vartheta_F(h, x)$ is the reachable set from X at time h of F. The reachable map $t \rightsquigarrow \vartheta_F(t, x)$ enjoys the semi-group property: $\forall t, s \geq 0, \quad \vartheta_F(t + s, x) = \vartheta_F(t, \vartheta_F(s, x))$.

Proposition 3.1 *The operator $K \mapsto \vartheta_F(h, K)$ is an algebraic dilation on $\mathcal{P}(X)$.*

Definition 3.2 [1, Definition 5.3.6.] *The set-valued map $\mathrm{Acc}_F(., t) : x \rightsquigarrow \mathrm{Acc}_F(x, t) = \bigcup_{s \leq t} \vartheta_F(x, s)$ is called the* accessibility map for F at t. *The* Accessibility tube *of x is the set-valued map $t \longrightarrow \mathrm{Acc}_F(x, t)$.*

Proposition 3.3 *Let F be a Marchaud set-valued map in X. The set-valued map $\mathrm{Acc}_F(t, .) : K \mapsto \mathrm{Acc}_F(K, t) = \cup_{x \in K} \mathrm{Acc}_F(x, t)$ is a dilation on $\mathcal{F}(X)$.*

Proof: For any family (K_i) of closed subsets of X,
$\mathrm{Acc}_F(., t)(\cup_i K_i) = \bigcup_{x \in \cup K_i} \mathrm{Acc}_F(x, t) = \cup_i \bigcup_{x \in K_i} \mathrm{Acc}_F(x, t) = \cup_i \mathrm{Acc}_F(., t)(K_i)$. □

3.2 Viability domain and viability kernel

Let K be a subset of the domain of F. A function $x(\cdot) : I \mapsto X$ is said to be viable in K on the interval $I \subset \mathbb{R}^+$ if and only if

$$\forall t \in I, \quad x(t) \in K$$

We shall say that K is locally viable under F (or enjoys the local viability property for the set-valued map F) if for any initial state x_0 in K, there exist $T > 0$ and a solution on $[0, T]$ to differential inclusion (2) starting at x_0 which is viable in K. It is said to be (globally) viable under F (or to enjoy the (global) viability property) if we can take $T = \infty$.
The notion of the viability of $x(.)$ is related to tangency of velocities $x'(.)$ to K. Let us make this more precise and introduce a tangent cone to K et $x \in K$, called the contingent cone, which generelizes the idea of tangent space in non regular framework.

Let $K \subset X$ be a subset of the vector space X, and $x \in \overline{K}$ belongs to the closure of K. The contingent cone $T_K(x)$ to K at x is defined by [4, Definition 4.1.1.]:

$$T_K(x) := \left\{ v \mid \liminf_{h \to 0^+} \frac{d(x + hv, K)}{h} = 0 \right\} \tag{4}$$

where $d(x, K) = \inf_{y \in K} d(x, y)$ and d is the distance on X (see fig. 1). In other words, v belongs to $T_K(x)$ if and only if there exist a sequence of $h_n > 0$ converging to 0^+ and a sequence of $v_n \in X$ converging to v such that $\forall n \geq 0,\ x + h_n v_n \in K$. We see obviously that $\forall x \in Int(K),\ T_K(x) = X$ where $Int(K)$ denotes the interior of K.

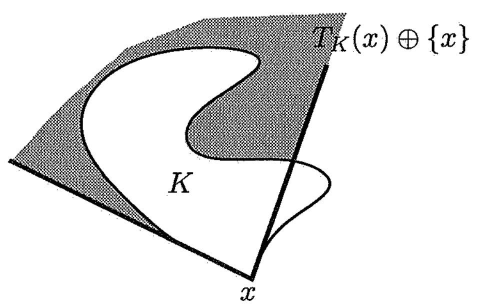

FIG. 1. – *Contingent cone of a subset K at x.*

Definition 3.4 (Viability Domain) [1, Definition 3.2.1] *Let $F : X \rightsquigarrow X$ be a nontrivial set-valued map. We shall say that a subset $K \subset \mathrm{Dom}(F)$ is a* viability domain *of F if and only if*

$$\forall x \in K, F(x) \cap T_K(x) \neq \emptyset$$

Theorem 3.5 (Viability Theorem) [1, Theorem 3.3.5.] *Consider a Marchaud map[4] $F : X \rightsquigarrow X$ and a closed subset $K \subset \mathrm{Dom}(F)$ of a finite dimensional vector space X.*

[4]We denote by $\|F(x)\| := \sup_{y \in F(x)} \|y\|$ and we say that F has linear growth if there exists a positive constant c such that: $\forall x \in \mathrm{Dom}(F), \|F(x)\| \leq c(\|x\| + 1)$. We shall say that F is a Marchaud map if it is upper semicontinuous, has compact convex images and linear growth.

If K is a viability domain, then for any initial state $x_0 \in K$, there exists a viable *solution on* $[0, \infty[$ *to differential inclusion (2).*

If a closed subset K is not a viability domain, the question arises whether there are closed viability subsets of K viable under F and even, whether there exists a largest closed subset of K viable under F.

Definition 3.6 (Viability Kernel) [1, Definition 4.1.1] *Let K be a subset of the domain of a set-valued map $F : X \rightsquigarrow X$. We shall say that the largest closed subset of K viable under F (which may be empty) is the* viability kernel of K for F *and denote it by* $\text{Viab}_F(K)$.

Assume that $F : X \rightsquigarrow X$ is Marchaud. Then, if $\Omega \subset X$ is an open subset $\text{Viab}_F(X \backslash \Omega)$ is the set of initial states x from which at least one solution never reaches Ω. It is a closed subset. We say that Ω is absorbing if this closed set is empty.

Its complement, called the absorbing domain, is the set

$$\text{Abs}_F(\Omega) = X \backslash \text{Viab}_F(X \backslash \Omega)$$

of initial states x from which all solutions reach Ω in finite time.

3.3 Invariance domain and invariance kernel

We shall say that a subset K is invariant under the differential inclusion (2) if for any initial state $x_0 \in K$, all solutions to the differential inclusion starting from it are viable in K. We then can prove the characterization of closed subsets invariant under a Lipschitz map:

Theorem 3.7 [1, Theorem 5.2.1.] *Let us assume that $F : X \rightsquigarrow X$ is Lipschitz with nonempty compact values. Then a closed subset $K \subset \text{Int}(\text{Dom}(F))$ is locally invariant under F if and only if K is an* invariance domain*:*

$$\forall x \in K, F(x) \subset T_K(x)$$

We then can introduce the concept of invariance kernels and invariance envelopes:

Definition 3.8 (Invariance Kernel, Invariance Envelope) [19, Definition 1.] *Let K be a closed subset. We shall say that the largest closed subset of K invariant under F (which may be empty) is the* invariance kernel of K for F *and denote it by* $\text{Inv}_F(K)$. *We shall say that the smallest closed subset invariant under F containing K is the* invariance envelope of K for F *and denote it by* $\text{Env}_F(K)$.

3.4 Exit Tubes

Let K be a closed subset of X and $x(\cdot) : [0, +\infty[\mapsto X$ be a continuous function. We denote by τ_K the exit functionnal associating with $x(\cdot)$ its exit time $\tau_K(x(\cdot))$ defined by [1, Definition 4.2.1] (See fig. 2):

$$\tau_K(x(\cdot)) := \inf\{t \in [0, +\infty[\mid x(t) \notin K\}$$

It is obvious that $\forall t \in [0, \tau_K(x(\cdot))[$, $x(t) \in K$, and if $\tau_K(x(\cdot))$ is finite then $x(\tau_K(x(\cdot))) \in \partial K$.

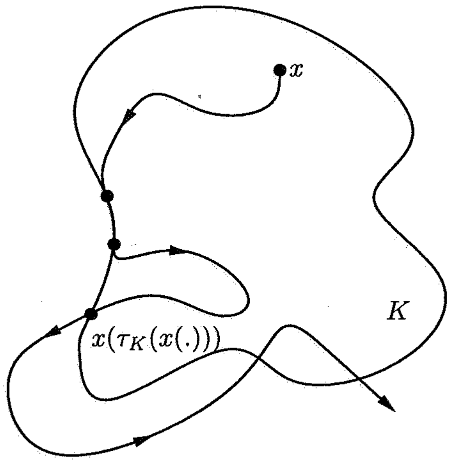

FIG. 2. – *Exit time.*

Then we can associate the function $\tau_K^\sharp : K \mapsto \mathbb{R}_+ \cup \{+\infty\}$ defined by: $\tau_K^\sharp : K \mapsto \mathbb{R}_+ \cup \{+\infty\}$ (resp. $\tau_K^\flat : K \mapsto \mathbb{R}_+ \cup \{+\infty\}$) defined by:

$$\tau_K^\sharp(x) := \sup_{x(\cdot)\in\vartheta_F(x)} \tau_K(x(\cdot)) \quad \text{(resp. } \tau_K^\flat(x) := \inf_{x(\cdot)\in\vartheta_F(x)} \tau_K(x(\cdot)))$$

called the **exit function** (resp. the **global exit function**).

Finally, we can associate with any $t \geq 0$ the two subsets :

$$\begin{array}{rcl} \mathrm{EXIT}_F(K,t) & := & \{x \in K \mid \tau_K^\sharp(x) \geq t\} \\ \mathrm{Exit}_F(K,t) & := & \{x \in K \mid \tau_K^\flat(x) \geq t\} \end{array}$$

We shall say that the set-valued map $t \mapsto \mathrm{Exit}_F(K,t)$ (resp. $t \mapsto \mathrm{EXIT}_F(K,t)$) is the **exit tube** (resp. the **global exit tube**).

One can prove [1] that *when F is Lipschitz with closed convex values, the graphs of these four tubes are closed.*
Furthermore $\mathrm{Exit}_F(K,t)$ *(resp.* $\mathrm{EXIT}_F(K,t)$*) is the subset of initial states $x \in K$ such that one solution (resp. all solutions) $x(\cdot)$ to differential inclusion* (3) *starting at x remains in K for all time in* $[0,t]$.

In summary, we have:

$$\begin{cases} \mathrm{Exit}_F(K,t) & = \vartheta_F^{-1}\{x(\cdot) \mid \forall s \in [0,t], x(s) \in K\} \\ \\ \mathrm{EXIT}_F(K,t) & = \vartheta_F^{+1}\{x(\cdot) \mid \forall s \in [0,t], x(s) \in K\} \end{cases}$$

When $t_1 \leq t_2$ then: $\mathrm{Exit}_F(K,t_2) \subseteq \mathrm{Exit}_F(K,t_1) \subseteq ... \subseteq \mathrm{Exit}_F(K,0) = K$.

In particular, for $t = +\infty$,

$$\text{Viab}_F(K) = \text{Exit}_F(K, +\infty) = \bigcap_{t\geq 0} \text{Exit}_F(K,t) \quad \text{and} \quad \text{Inv}_F(K) = \bigcap_{t\geq 0} \text{EXIT}_F(K,t)$$

Let $F : X \rightsquigarrow X$ be a set-valued map. The operator $K \mapsto Exit_F(K,t)$ is an increasing antiextensive operator.

Proposition 3.9 *Let $F : X \rightsquigarrow X$ be a set-valued map. The operator $EXIT_F(.,t) : K \mapsto \text{EXIT}_F(K,t)$ is an algebraic erosion on $\mathcal{F}(X)$.*

Proof: Since $EXIT_F(.,t)(K) = \text{EXIT}_F(K,t) = \{x \in K \mid \forall s \leq t,\ \vartheta_F(s,x) \subset K\}$, it is obvious that: $EXIT_F(.,t)(\cap K_i) = \text{EXIT}_F(\cap K_i, t) = \cap\text{EXIT}_F(K_i,t) = \cap EXIT_F(.,t)(K_i)$. □

Let $F : X \rightsquigarrow X$ be a Marchaud map, then $K \mapsto \vartheta_F(h,K)$ is $\uparrow$-continuous, since this operator is an algebraic dilation on $\mathcal{P}(X)$, and $K \mapsto EXIT_F(.,t)(K) = EXIT_F(K,t)$ is $\downarrow$-continuous as an algebraic erosion on $\mathcal{F}(X)$. Since $K \mapsto EXIT_F(.,t)(K)$ is increasing and $\downarrow$-continuous, we deduce that

Proposition 3.10 *Let $F : X \rightsquigarrow X$ be a Marchaud map. The operator $EXIT_F(.,t) : K \mapsto EXIT_F(K,t)$ is u.s.c. on $\mathcal{F}(X)$.*

3.5 Accessibility tube

Let $F : X \rightsquigarrow X$ be a Marchaud set-valued map. Let $\vartheta_F(h,x)$ be the set of the values $x(h)$ at time h of the solutions x of (3). We recall that the Accessibility tube of x is the set-valued map $t \rightarrow \text{Acc}_F(x,t) = \bigcup_{s\leq t} \vartheta_F(x,s)$.

Proposition 3.11 [15, Proposition 6.1.]

– *Let F be a Lispchitz set-valued map with compact convex values in X, then:*

$$Exit_F(K,h) \subseteq \text{Acc}_F(.,h)^{-1}(K)$$

– *Let F be a Marchaud set-valued map in X, then $EXIT_F(K,h) = \text{Acc}_F(.,h)^{+1}(K)$.*

Proof:

$$\begin{aligned}
Exit_F(K,h) &= \{x \in K \mid \sup_{x(\cdot)\in\vartheta_F(x)} \tau_K(x(\cdot)) \geq t\} \\
&= \{x \in K \mid \exists x(\cdot) \in \vartheta_F(x),\ \tau_K(x(\cdot)) \geq t\} \\
&= \{x \in K \mid \exists x(\cdot) \in \vartheta_F(x),\ \forall t \leq h,\ x(t) \in K\} \\
&\subseteq \{x \in K \mid \forall t \leq h,\ \exists x(\cdot) \in \vartheta_F(x),\ x(t) \in K\} \\
&\subseteq \{x \in X \mid \forall t \leq h,\ \ \vartheta_F(t,x) \cap K \neq \emptyset\} \\
&\subseteq \{x \in X \mid \bigcup_{t\leq h} \vartheta_F(t,x) \cap K \neq \emptyset\} \\
&\subseteq \text{Acc}_F(.,h)^{-1}(K)
\end{aligned}$$

$$
\begin{aligned}
EXIT_F(K,h) &= \{x \in K \mid \inf_{x(\cdot)\in\vartheta_F(x)} \tau_K(x(\cdot)) \geq t\} \\
&= \{x \in K \mid \forall x(\cdot) \in \vartheta_F(x),\ \tau_K(x(\cdot)) \geq t\} \\
&= \{x \in K \mid \forall x(\cdot) \in \vartheta_F(x),\ \forall t \leq h,\ x(t) \in K\} \\
&= \{x \in K \mid \forall t \leq h,\ \forall x(\cdot) \in \vartheta_F(x),\ x(t) \in K\} \\
&= \{x \in X \mid \forall t \leq h,\ \vartheta_F(t,x) \subset K\} \\
&= \{x \in X \mid \bigcup_{t\leq h} \vartheta_F(t,x) \subset K\} \\
&= \mathrm{Acc}_F(.,h)^{+1}(K)
\end{aligned}
$$

□

Theorem 3.12 *Let F be a Marchaud set-valued map in X. Let us consider the two operators on $\mathcal{F}(X)$: $EXIT_F(.,t) : K \mapsto EXIT_F(K,t)$ and $\mathrm{Acc}_F(.,t) : K \mapsto \mathrm{Acc}_F(K,t)$. Then $EXIT_F(.,t)^{\circledast} = \mathrm{Acc}_F(.,t)$ on $\mathcal{F}(X)$.*

Proof: For any $K, H \in \mathcal{F}(X)$

$$
\begin{aligned}
\mathrm{Acc}_F(.,t)(K) \subset H &\Leftrightarrow \bigcup_{s\leq t} \vartheta_F(s,K) \subset H \\
&\Leftrightarrow \forall x \in K,\ \forall s \leq t,\ \vartheta_F(s,x) \subset H \text{ (In particular } x \in H) \\
&\Leftrightarrow K \subset EXIT_F(H,t) = \{x \in H \mid \forall s \leq t,\ \vartheta_F(s,x) \subset H\} \\
&\Leftrightarrow K \subset EXIT_F(.,t)(H)
\end{aligned}
$$

□

It follows from proposition 2.7 that :

Corollary 3.13 *The operator $K \mapsto \mathrm{Acc}_F(EXIT_F(K,t),t)$ is an algebraic opening and the operator $K \mapsto EXIT_F(\mathrm{Acc}_F(K,t),t)$ is an algebraic closing on $\mathcal{F}(X)$.*

3.6 Properties of the viability kernel

Let us consider a Marchaud map $F : X \rightsquigarrow X$. Let Viab_F be the following operator on $\mathcal{F}(X)$ defined by $\mathrm{Viab}_F : K \mapsto \mathrm{Viab}_F(K)$. In this section, we will study some properties of this operator.

3.6.1 Algebraic properties

We first recall a characterization of the viability kernel.

Theorem 3.14 [2, Theorem 4.1.15.] *Let $F : X \rightsquigarrow X$ be a Marchaud map and $K \subset \mathrm{Dom}(F) = \{x \in X \mid F(x) \neq \emptyset\}$ be closed. Then the viability kernel of K exists (possibly empty) and is equal to the subset of initial states such that at least one solution starting from them is viable in K:*

$$\mathrm{Viab}_F(K) = \vartheta_F^{-1}\{x(\cdot) \mid \forall t \geq 0, x(t) \in K\}$$

Proposition 3.15 *The operator $\mathrm{Viab}_F : K \mapsto \mathrm{Viab}_F(K)$ is an algebraic opening on $\mathcal{F}(X)$.*

Proof:

- By theorem 3.14, Viab_F is idempotent.
- It is obvious that Viab_F is anti-extensive because $\forall K \in \mathcal{F}(X),\ \ \mathrm{Viab}_F(K) \subset K$.
- Viab_F is increasing: Suppose that $K \subset L$, $\forall x \in \mathrm{Viab}_F(K)$, $\exists x(.)$ such that $\forall t \in [0,T]$, $x(t) = \vartheta_F(t,x)$ and $x(t) \in K \subset L$, this implies $x \in \mathrm{Viab}_F(L)$, then we deduce that $\mathrm{Viab}_F(K) \subset \mathrm{Viab}_F(L)$.

□

It follows from proposition 2.8 that:

Corollary 3.16 *The operator $\mathrm{Abs}_F(.)$ defined by $\mathrm{Abs}_F(\Omega) = X \backslash \mathrm{Viab}_F(X \backslash \Omega)$ is an algebraic closing on the space $\mathcal{F}^*(X)$ of all open subsets of X.*

Since Viab_F is increasing, for all family (K_i), we have:

$$\begin{array}{rcl} \mathrm{Viab}_F(\cap_i K_i) & \subset & \cap_i \mathrm{Viab}_F(K_i) \\ \mathrm{Viab}_F(\overline{\cup_i K_i}) & \supset & \overline{\cup_i \mathrm{Viab}_F(K_i)} \end{array}$$

Let $F : X \rightsquigarrow X$ be a Lipschitz and Marchaud set-valued map. Then from proposition 2.14 we deduce that:

$$\forall K \in \mathcal{F}(X),\ \forall t \geq 0,\ \ \mathrm{Viab}_F(Exit_F(t,K)) = Exit_F(t, \mathrm{Viab}_F(K)) = \mathrm{Viab}_F(K)$$

3.6.2 Continuity property

Proposition 3.17 [1, Corollary 4.1.5.] *Let us consider a set-valued map $F : X \rightsquigarrow X$ satisfying uniform linear growth and an arbitrary sequence of closed sets K_n.*
Then $\mathsf{Limsup}_{n\to\infty} \mathrm{Viab}_F(K_n) \subset \mathrm{Viab}_F(\mathsf{Limsup}_{n\to\infty} K_n)$.

From this proposition, it is obvious that Viab_F is a $\downarrow$-continuous map. Since Viab_F is increasing, from proposition 2.23 we have:

Corollary 3.18 *Let $F : X \rightsquigarrow X$ be a Marchaud map. Let $\mathrm{Viab}_F : \mathcal{F}(X) \mapsto \mathcal{F}(X)$ be defined $K \mapsto \mathrm{Viab}_F(K)$ then Viab_F is u.s.c..*

3.6.3 Set of fixpoints

From proposition 3.15, we deduce that $\mathrm{Viab}_F(K) = \cup\{L \in \mathcal{F}(X) \mid L \in \mathrm{Fix}(\mathrm{Viab}_F) \quad L \subset K\}$ where $\mathrm{Fix}(\mathrm{Viab}_F)$ is the set of all viability domains for F. This set is a complete lattice. The domain of the solution map ϑ_F is the largest closed viability domain contained in the domain of F, and:

$$\begin{aligned} \bigwedge \mathrm{Fix}(\mathrm{Viab}_F) &= \emptyset \\ \bigvee \mathrm{Fix}(\mathrm{Viab}_F) &= \mathrm{Dom}(\vartheta_F) \end{aligned}$$

Definition 3.19 (Limit set) [1, Definition 3.7.1] *Let $x(\cdot)$ be a solution to differential inclusion (2). We say that the subset*

$$\omega_F(x(\cdot)) := \bigcap_{T>0} cl(x([T,\infty[)) = \limsup_{t\to+\infty}\{x(t)\}$$

of its cluster points when $t \to \infty$ is the limit set *of $x(\cdot)$. If K is a subset of* $\mathrm{Dom}(\vartheta_F)$ *and $\vartheta_F(.,K)$ the reachable map, we denote by $\omega_F(K) := \limsup_{t\to+\infty}\vartheta_F(t,K)$ the ω-limit set of the subset K.*

Proposition 3.20 (Limit sets are viability domains) [1, Theorem 3.7.2] *Let us consider a Marchaud map $F : X \rightsquigarrow X$. Then the limit sets of the solutions to differential inclusion (2) are closed viability domains which are connected when $\omega_F(x(\cdot))$ are compact. In particular, the limits of solutions to differential inclusion (2), when they exist, are equilibria of F and the trajectories of periodic solutions to the differential inclusion (2) are also closed viability domains.*

Proposition 3.21 *Let $F : X \rightsquigarrow X$ be a Marchaud map. Then for all $x \in K$ and for all $x(.) \in \vartheta_F(x) \cap K$:*

$$\mathbb{E}_F(x(.)) = \left(\bigcup_{t\geq 0}\{x(t)\}\right) \cup \omega_F(x(.))$$

is a closed viability domain and,

$$\mathrm{Viab}_F(K) = \bigcup_{x\in K}\left(\bigcup_{x(.)\in\vartheta_F(x)\cap K} \mathbb{E}_F(x(.))\right)$$

Proof: $\forall x \in \mathrm{Viab}_F(K)$, the subset $\{x\}$ is contained into a minimal closed viability domain (Zorn's lemma for the inclusion order on the family of non empty closed viability domains of F). Minimal domains which contain x are $V(x) = \bigcup_{x(.)\in\vartheta_F(x)\cap K}\mathbb{E}_F(x(.))$. It is clear that $\mathrm{Viab}_F(K) = \bigcup_{x\in K} V(x)$.

- Let us consider, for all $x \in K$ and for all $x(.) \in \vartheta_F(x) \cap K$, the following set $\mathbb{E}_F(x(.)) = \left(\bigcup_{t\geq 0}\{x(t)\}\right) \cup \omega_F(x(.))$, where $x(.) \in \vartheta_F(x) \cap K$. The set $\mathbb{E}_F(x(.))$ is a viability domain because $\forall y \in \mathbb{E}_F(x(.))$, we have $y = x(t)$ or $y = \lim_{t_n\to\infty} x(t_n)$ then $y(.) = x(. - t)$ or $y(.) = x(. - t_n)$ is a solution viable in $\mathbb{E}_F(x(.))$.
- $\forall x \in V(x)$, $\exists x(.) \in \vartheta_F(x)$ viable in $V(x)$, then $\forall t \geq 0$, $x(t) \in V(x)$. We deduce that $\mathbb{E}_F(x) \subset V(x)$. Since $\mathbb{E}_F(x(.))$ is a viability domain for one of the solutions of the differential inclusion (3) and $V(x)$ is a minimal closed viability domain contain in K, we have $V(x) = \bigcup_{x(.)\in\vartheta_F(x)\cap K}\mathbb{E}_F(x(.))$.

- Since $\forall y \in \mathbb{E}_F(x(.)) \Rightarrow \mathbb{E}_F(y(.)) \subset \mathbb{E}_F(x(.))$, we have a preorder relationship $x(.) \succ y(.) \Rightarrow \mathbb{E}_F(y(.)) \subset \mathbb{E}_F(x(.))$. Therefore for every point $x \in K$, the subset $\{x\}$ is contained in a minimal element $\mathbb{E}_F(x(.))$ for this preorder.

□

We can deduce that :

$$\text{Viab}_F(K) = \bigcup\{E \in \mathcal{E} \mid E \subset K\} \text{ where } \mathcal{E} = \bigcup_{x \in X} \bigcup_{x(.) \in \vartheta_F(x)} \mathbb{E}_F(x(.))$$

3.7 Properties of the invariance kernel and invariance envelope

Let us consider a Marchaud map $F : X \rightsquigarrow X$. Let Inv_F be the following operator on $\mathcal{F}(X)$ defined by $K \mapsto \text{Inv}_F(K)$. In this section, we will study some algebraic properties of this operator.
Let us recall a useful characterization of the invariance kernel:

Theorem 3.22 [2, Theorem 4.2.6.] *Let us consider a Lipschitz map $F : X \rightsquigarrow X$ with closed values. Let K be closed. Then the invariance kernel of K exists (possibly empty) and is equal to the subset of initial states such that all solutions starting from them is viable in K:*

$$\text{Inv}_F(K) = \vartheta_F^{+1}\{x(\cdot) \mid \forall t \geq 0, x(t) \in K\}$$

From theorem 3.22, it is easy to deduce that:

Proposition 3.23 *The operator $\text{Inv}_F : K \mapsto \text{Inv}_F(K)$ is an increasing algebraic erosion and an algebraic opening on $\mathcal{F}(X)$.*

It is clear that $\text{Inv}_F(K_1 \cap K_2) = \text{Inv}_F(K_1) \cap \text{Inv}_F(K_2)$ and more generally, that the invariance kernel of any intersection of closed subsets K_i $(i \in I)$ is the intersection of invariance kernels of the K_i.
It follows that:

Proposition 3.24 *The operator $\text{Inv}_F : K \mapsto \text{Inv}_F(K)$ is u.s.c. on $\mathcal{F}(X)$.*

Proposition 3.25 [1, Proposition 5.4.5.] *Let us assume that the solution map ϑ_F is lower semicontinuous from Ω to $\mathcal{C}(0, \infty; X) = \{f : [0, \infty] \mapsto X$ which is continuous$\}$. Then the lower limit of closed subsets $K_n \subset \Omega$ invariant under F is also invariant under F. In particular, the lower limit of the invariance kernels of a sequences of closed subsets $K_n \subset \Omega$ contains the invariance kernels of the lower limit of the sequence K_n:*

$$\operatorname*{Liminf}_{n \to \infty}(\text{Inv}_F(K_n)) \supset \text{Inv}_F\left(\operatorname*{Liminf}_{n \to \infty}(K_n)\right)$$

It follows that if the solution map ϑ_F is lower semicontinuous, then the operator $\text{Inv}_F : K \mapsto \text{Inv}_F(K)$ is $\uparrow$-continuous. Since it is an algebraic erosion, it is $\downarrow$-continuous. We deduce that, under the previous assumptions, the operator $\text{Inv}_F : K \mapsto \text{Inv}_F(K)$ is continuous.

Proposition 3.26 [19, Proposition 2.] *Assume that $F : X \rightsquigarrow X$ is Lipschitz with non empty closed values. Then the invariance envelope and the accessibility map are related by:*

$$\mathrm{Env}_F(K) = \overline{\mathrm{Acc}_F(K, +\infty)}$$

Furthermore, if we suppose that $K = \overline{Int(K)}$ then:

$$\mathrm{Env}_F(K) = \overline{X \setminus \mathrm{Inv}_{-F}\hat{K}} \quad \textit{where } \hat{K} = \overline{X \setminus K}$$

Let $F : X \rightsquigarrow X$ be Lipschitz with non empty closed values. Let K, H be two non empty closed subsets of X, then:

- if $H \subset K$ then $\mathrm{Env}_F(H) \subset \mathrm{Env}_F(K)$,
- $\mathrm{Env}_F(H \cap K) \subset \mathrm{Env}_F(H) \cap \mathrm{Env}_F(K)$,
- the subset K is invariant if and only if $K = \mathrm{Env}_F(K)$.

It follows that:

Proposition 3.27 *The operator $\mathrm{Env}_F : K \mapsto \mathrm{Env}_F(K)$ is an algebraic dilation and closing on $\mathcal{F}(X) \cap \{K = \overline{Int(K)}\}$.*

Since the intersection of two invariance domains is still an invariance domain, the invariance envelope $\mathrm{Env}_F(K)$ is defined as the intersection of all closed subsets containing K. Then we obtain: $\mathrm{Env}_F(K) = \cap\{B \in \mathcal{B} \mid K \subset B\}$ where $\mathcal{B}$ is the family of all invariant domains under F.

4 Application to morphological operators

Let B be a subset of a topological vector space X. We consider the multivalued map T_B defined by: $T_B(x) = B_x = \{x + b \text{ where } b \in B\} = B \oplus \{x\}$. We put $\check{B} = -B = \{-b \mid b \in B\}$ the symetrical set of B.

We recall [12] that if K and B are two subsets of X,

- The morphological dilation of K by B is defined by: $K \oplus \check{B} = \{x \mid B_x \cap K \neq \emptyset\}$.
- The morphological erosion of K by B is defined by: $K \ominus \check{B} = \{x \mid B_x \subset K\}$.
- The morphological opening of K by B is defined by: $K_B = (K \ominus \check{B}) \oplus B = \cup_x\{B_x \mid B_x \subset K\}$.
- The morphological closing of K by B is defined by: $K^B = (K \oplus \check{B}) \ominus B$.

It is obvious that the morphological dilation (resp. erosion, opening, closing) is an algebraic dilation (resp. erosion, opening, closing), and we have:

$$T_B^{-1}(K) = \{x \mid B_x \cap K \neq \emptyset\} = K \oplus \check{B} \quad \text{and} \quad T_B^{+1}(K) = \{x \mid B_x \subset K\} = K \ominus \check{B}$$

From formula (2.8), it follows that: $M_K^{+1}(X^c) = M_K^{-1}(X)^c \Leftrightarrow X^c \oplus \check{K} = \left(X \ominus \check{K}\right)^c$, then the operators $K \mapsto K \oplus \check{B}$ and $K \mapsto K \ominus \check{B}$ are duals.

We observe at once that, if F is the constant set-valued map $y \rightsquigarrow F(y) = B$ where B is a compact convex set, then $\vartheta_F(h, K) = K \oplus hB$ is the Minkowski addition of K by the homothetic of B at scale h. In fact, we have: $\vartheta_F(h, x) = \{x(h) \mid x'(t) \in F(x(t)) \; \& \; x(0) = x\}$. Then, differential inclusion (3) is equivalent to $x'(t) \in B$ and $x(0) = x$, i.e. $x(h) = x + \int_0^h b(t)dt$ for $b(t) \in B$. Under the convexity assumption of B, we deduce that $\vartheta_F(h, x) = \{x\} \oplus hB$, and we finally obtain $\vartheta_F(h, K) = \cup_{x \in K}\{\vartheta_F(h, x)\} = \cup_{x \in K}\{x\} \oplus hB = K \oplus hB$.

We say that B is a **barrel** set of the finite dimensional vector space X if B is a convex, symmetric (i.e. $-B = B$), compact set with non-empty interior. We can equip the space X with a metric derived from a norm associated with a barrel set B defined by: $\|\cdot\|_B$: $\|x\|_B = \min\{\lambda, \; \lambda \geq 0; \; x \in \lambda B\}$ and the induced distance d^B is defined by: [21, Chapter 9.]

$$d^B(x, y) = \|x - y\|_B \Leftrightarrow d^B(x, y) = \min\{\lambda, \; \lambda \geq 0; \; y \in \{x\} \oplus \lambda B\} \tag{5}$$

i.e. $d^B(x, y)$ is the size of the biggest homothetic set of B centered on x and containing y. It is obvious that in the plane $\mathbb{R}^2$, the Euclidean norm, the L_1-norm, i.e. $\|x\|_1 = \mid x_1 + x_2 \mid$, and the L_∞-norm, i.e. $\| x\|_\infty = \sup(|x_1|, |x_2|)$ where $x = (x_1, x_2)$, are $\|\cdot\|_B$ for B respectively be a disk, a diamond and a square.

Let us now, consider the following differential inclusion: $x'(t) \in F(x(t))$ where F is the constant set-valued map equal to the barrel B, i.e. we consider the differential inclusion $x'(t) \in B$, then $\vartheta_B(h, x) = \{x\} \oplus hB$, and we have:

Proposition 4.1 *Let K be a closed subset of X and B be a compact convex subset of X containing the origin. Let $\varphi : X \to \mathbb{R}^+$ be a 1-Lipschitz single valued map, and let be $F(y) = \varphi(y)B$ for all $y \in X$. Then, for all initial state x, we have:*

$$\mathrm{Acc}_B(x, t) = \vartheta_B(t, x) = \{x\} \oplus \sup_{x(\cdot) \in \vartheta_B(x)} \left(\int_0^t \varphi(x(s))ds \right) B \tag{6}$$

<u>Proof</u>: We observe that $\vartheta_B(h, x) = \{x\} \oplus \sup_{x(\cdot) \in \vartheta_B(x)} \left(\int_0^h \varphi(x(s))ds \right) B$ is the set of solutions of the differential inclusion $x'(t) \in \varphi(x(t))B$ with initial state $x(0) = x$. Under the assumption $O \in B$, we have: For $a < b$, $\left(\int_0^a \varphi(x(s))ds \right) B \subset \left(\int_0^b \varphi(x(s))ds \right) B$, and finally:

$$\bigcup_{h \leq t} \sup_{x(\cdot) \in \vartheta_B(x)} \left(\int_0^h \varphi(x(s))ds \right) B = \sup_{x(\cdot) \in \vartheta_B(x)} \left(\int_0^t \varphi(x(s))ds \right) B$$

□

Lemma 4.2 *Let B be a barrel and K be a compact subset of X such that $\overline{Int(K)} = K$, then we have:*

$$\forall K \in X, \quad \tau_K^\flat(x) = d^B(x, K^c)$$

where $d^B(x, y) = \|x - y\|_B$, and $d^B(x, Y) = \inf_{y \in Y} d^B(x, y)$ is the distance to K^c function associated with B.

From lemma 4.2, we deduce that:

Proposition 4.3 *Let B be a barrel and K be a compact subset of X such that $\overline{Int(K)} = K$, then we have:*

$$\begin{cases} Exit_B(K,h) & = K & \\ \vartheta_B(h,K) & = \vartheta_B(h,.)^{-1}(K) & = K \oplus hB \\ EXIT_B(K,h) & = \vartheta_B(h,.)^{+1}(K) & = K \ominus hB \\ \vartheta_B(h, EXIT_B(K,h)) & = (K \ominus hB) \oplus hB & \end{cases}$$

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