Watersheding hierarchies

Deise Santana Maia
Jean Cousty, Laurent Najman, Benjamin Perret

Université Paris-Est, Laboratoire d'Informatique Gaspard-Monge, ESIEE Paris

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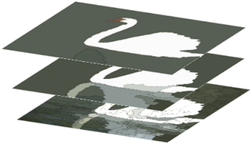
Hierarchical watersheds

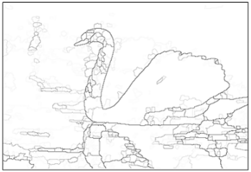
- Sequences of nested partitions such that each partition is a watershed of a filtering of the initial relief/map according to regional attributes, *e.g.*, on geometric, photometric, or learned information
- Hierarchies of segmentation can be equivalently represented by saliency maps

Original image

Hierarchical watershed $\mathcal{H}$



Saliency map of $\mathcal{H}$



Hierarchical watersheds

- Hierarchical watersheds are linked to a broader family of combinatorial optimization problems: minumum spanning trees, random walkers and graph cuts [C09]
- They satisfy interesting mathematical properties (e.g. they preserve a minimum spanning tree of the original graph)
- They are useful in object detection and perform well on natural images [P18]

[C09] C. Couprie, L. Grady, L. Najman, and H. Talbot. Power watersheds: A new image segmentation framework extending graph cuts, random walker and optimal spanning forest.

[P18] B. Perret, J Cousty, S. Guimaraes, D. Maia. Evaluation of hierarchical watersheds.

Watersheding hierarchies

Problem

Given a hierarchy $\mathcal{H}$ and a weighted graph (G, w)

- *find a hierarchical waterhsed $\mathcal{H}'$ of (G, w) which "approximates" $\mathcal{H}$*

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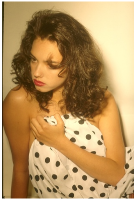
Contributions

Introduction of the watersheding operator

- *which transforms any hierarchy into a hierarchical watershed, and*
- *whose fixed points are the hierarchical watersheds*

Watersheding: applications

- Refinement of coarse hierarchies
    - *e.g.*, high quality hierarchical segmentation method (COB) [K18]


Original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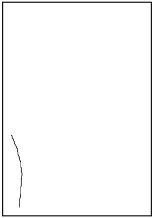
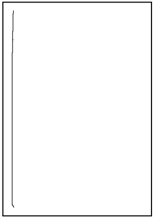

COB


Watersheding

Play/Pause Restart

[K18] K.-K. Maninis, J. Pont-Tuset, P. Arbeláez, and L. Van Gool. Convolutional oriented boundaries: From image segmentation to high-level tasks. PAMI. 2018.

Outlines

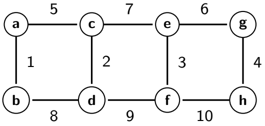
1 Hierarchical watersheds

2 Watersheding transform

3 Conclusion and perspectives

Graph settings

- (G, w) is an edge-weighted graph
  - which can be a pixel-adjacency graph
  - edge weights can represent a gradient of intensity



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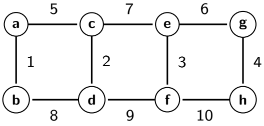

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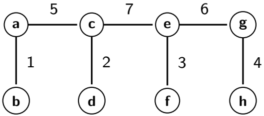

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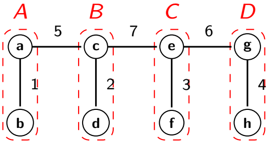
- For simplification, we consider that *the edges of G have pairwise distinct weights for w, which implies that (G, w) has a single minimum spanning tree (MST)*
- $\mathcal{S} = (M_0, \ldots, M_\ell)$ is any sequence of the minima of w ranked by importance according to some regional attribute like extinction values



Hierarchical watersheds

- A *hierarchical watershed of* (G, w) *for* $\mathcal{S}$ is a hierarchy of partitions $(\mathbf{P}_0, \ldots, \mathbf{P}_\ell)$ such that, for any $i \in \{0, \ldots, \ell\}$:
  - $\mathbf{P}_i$ is the connected component partition of a minimum spanning forest rooted in the minima ranked after i

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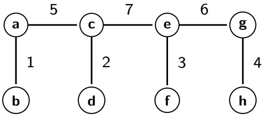

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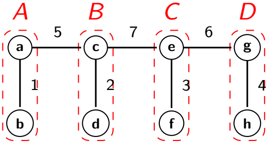

Minima of w

$$\mathcal{S} = (A, C, B, D)$$

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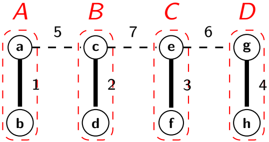

$\{a, b\}$ $\{c, d\}$ $\{e, f\}$ $\{g, h\}$ $\mathbf{P}_0$

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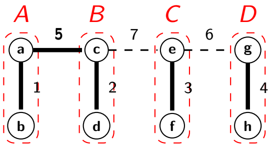

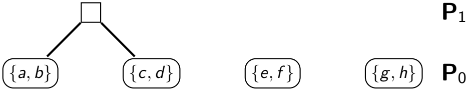

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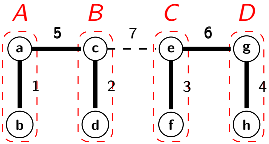

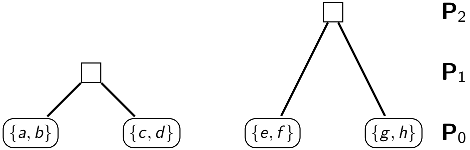

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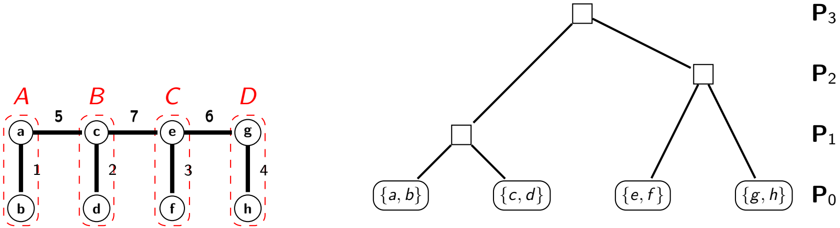

$(\mathbf{P}_0, \mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3)$:
hierarchical watershed of (G, w) by $\mathcal{M} = (A, C, B, D)$

Hierarchical watersheds

Definition

We say that $\mathcal{H}$ is a *hierarchical watershed of* (G, w) if there is a sequence $\mathcal{S}$ of minima such that $\mathcal{H}$ is the hierarchical watershed of (G, w) for $\mathcal{S}$

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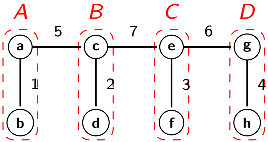
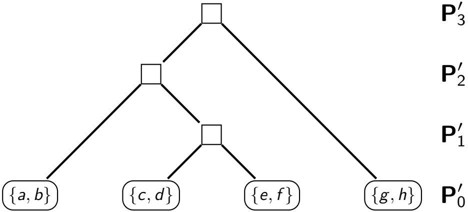
Remark 1: there are hierarchies which are not hierarchical watersheds of (G, w), e.g., $(\mathbf{P}'_0, \mathbf{P}'_1, \mathbf{P}'_2, \mathbf{P}'_3)$ *is* **not** *a hierarchical watershed of* (G, w)



(G, w)



$\mathcal{H}'$

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Remark 2: *a hierarchy* $\mathcal{H}$ *can be the hierarchical watershed of* (G, w) *for several sequences of minima of* w. *An algorithm to count the number of sequences of minima associated to a hierarchical watershed is provided in our companion paper "On the probabilities of hierarchical watersheds" (poster section)*

Outlines

1. Hierarchical watersheds

2. Watersheding transform

3. Conclusion and perspectives

Watersheding: general idea

- Intuitive idea: invert the algorithm to compute hierarchical watersheds introduced in [C13, N13]

Watersheding algorithm

Input: *weighted graph (G, w) and saliency map s of a hierarchy*
Output: *a saliency map $\omega(s)$, which is the watersheding of s*

1. *compute the* binary partition hierarchy (by altitude ordering) *of w*
2. *compute the* approximated extinction value (map) ξ_s *for s*
3. *compute the* estimated sequence of minima $\mathcal{S}_s$ *for s*
4. *return the saliency map of the hierarchical watershed of (G, w) for $\mathcal{S}_s$*

[C13] J. Cousty, L. Najman, B. Perret. Constructive links between some morphological hierarchies on edge-weighted graphs. ISMM 2013.

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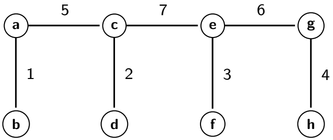
Binary partition hierarchy

Watersheding transform:
⟹ 1. binary partition hierarchy
2. approximated extinction map
3. estimated sequence of minima



(G, w)

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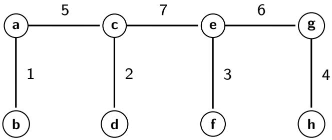

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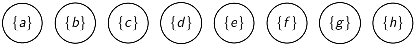

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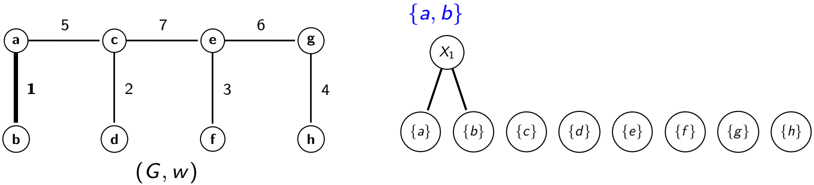

Any non-leaf region R of $\mathcal{B}$ can be mapped to an edge u_R of G which is then called the building edge of R (shown in blue)

Binary partition hierarchy

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$\Longrightarrow$ 1. binary partition hierarchy
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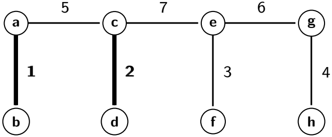

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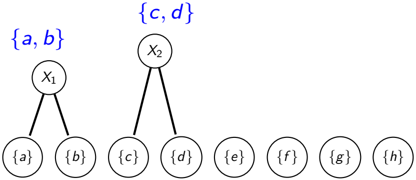

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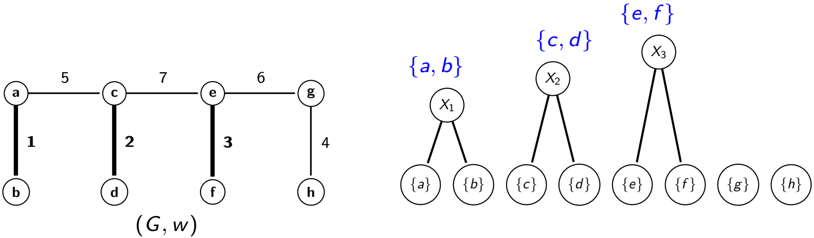

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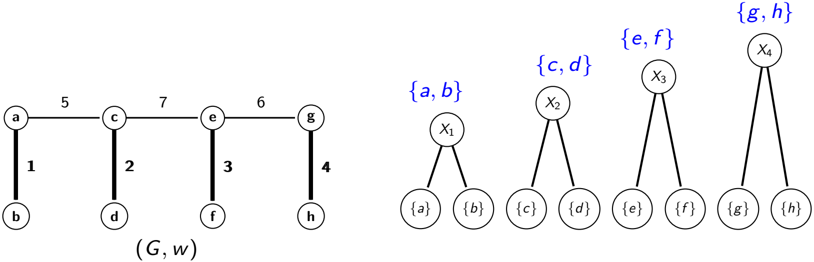

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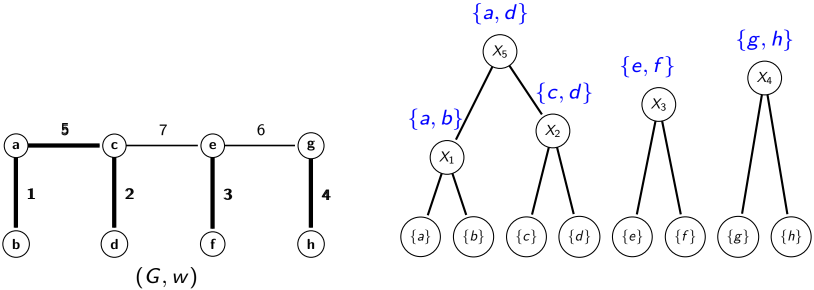

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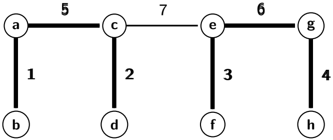

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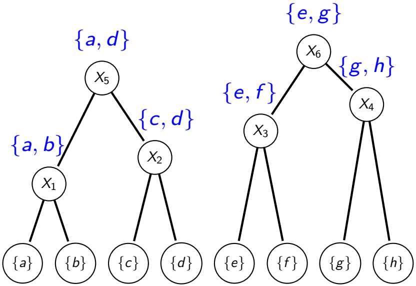

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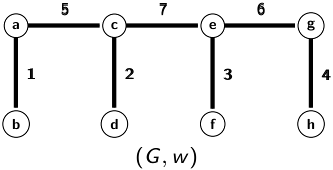

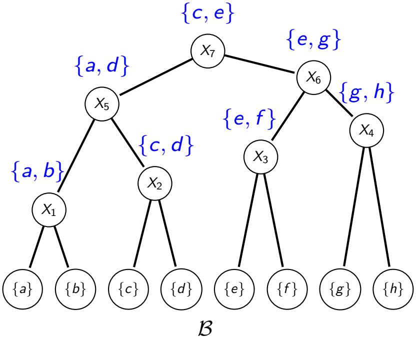

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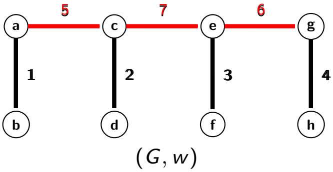

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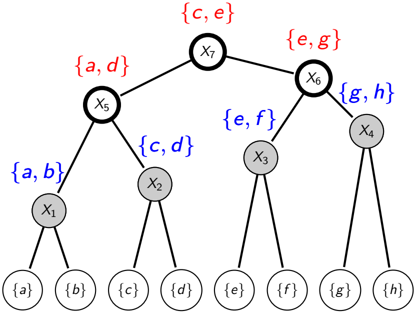

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Minima (in grey) and watershed-cut edges (in red)

Approximated extinction map (intuitive idea)

Watersheding transform:
1. binary partition hierarchy
⟹ 2. approximated extinction map
3. estimated sequence of minima

Definition (Extinction map for a sequence of minima)

Given a sequence $\mathcal{S} = (M_1, \dots, M_n)$ of minima of w, the extinction map for $\mathcal{S}$ is a map ϵ such that, for each region R of $\mathcal{B}$:

- $\epsilon(R)$ is zero if there is no minimum of w included in R
- otherwise, $\epsilon(R)$ is the maximum i such that $M_i \subseteq R$

As established in [C13], the saliency map of a hierarchical watershed for $\mathcal{S}$ can be efficiently computed from the extinction map for $\mathcal{S}$ by one pass on $\mathcal{B}$.

Definition (Estimated extinction map problem)

Given the saliency map s of a hierarchy $\mathcal{H}$, find a map ϵ_s, called an approximated extinction map of s such that, if $\mathcal{H}$ is a hierarchical watershed of (G, w), then this map is an extinction map which induces the hierarchical watershed $\mathcal{H}$.

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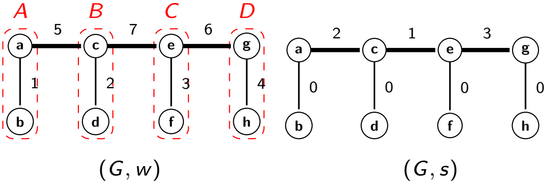
- Dominant regions:

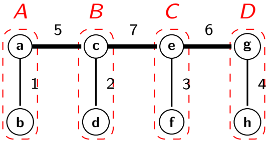


Approximated extinction map

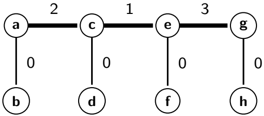
Watersheding transform:

1. binary partition hierarchy
2. ⇒ approximated extinction map
3. estimated sequence of minima

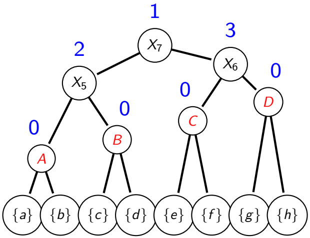
- Dominant regions:



(G, w)



(G, s)



$\mathcal{B}$ and s

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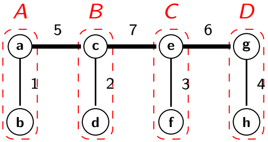

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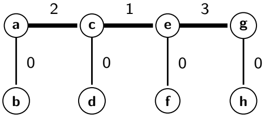

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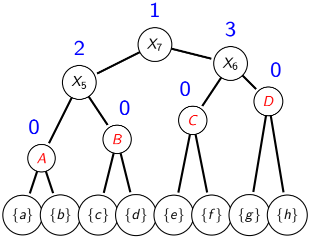

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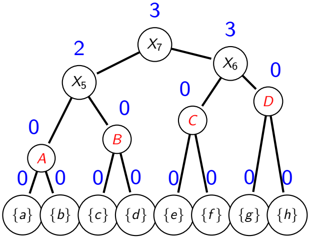

$\vee^s$

$\vee^s$: suppremum weights among the regions included in each region of $\mathcal{B}$

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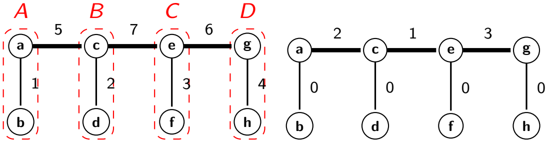

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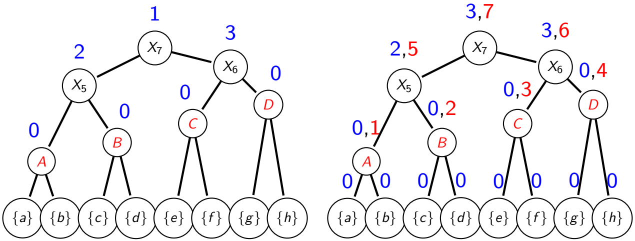

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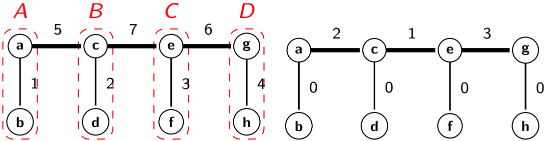

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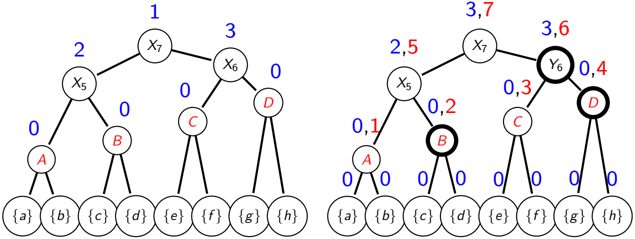

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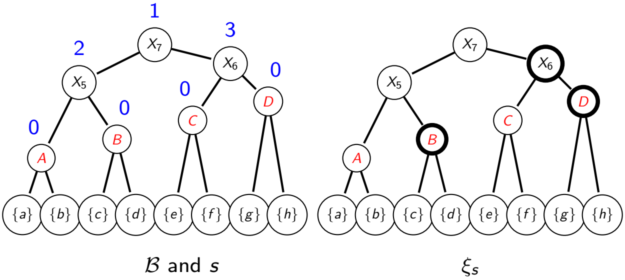

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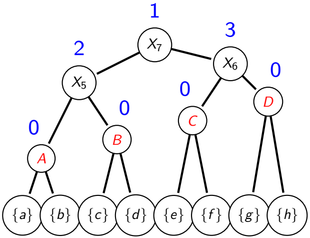

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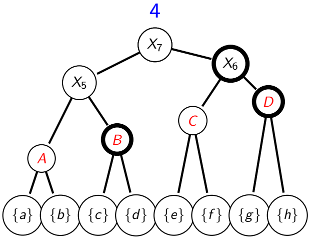

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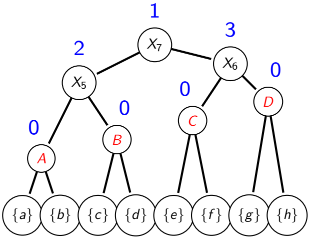

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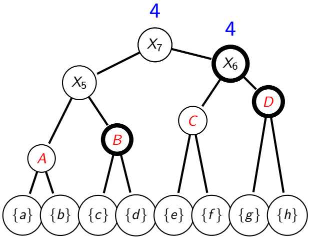

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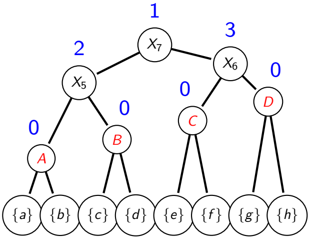

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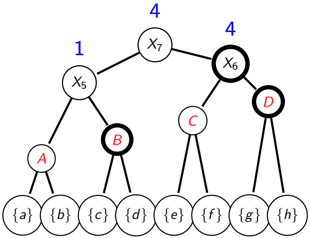

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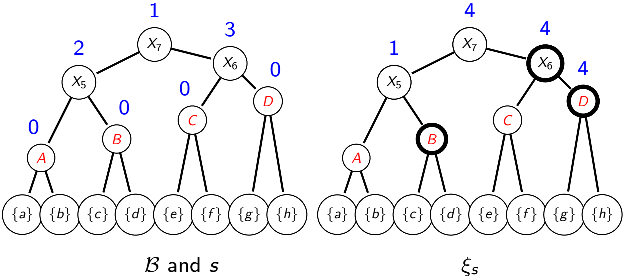

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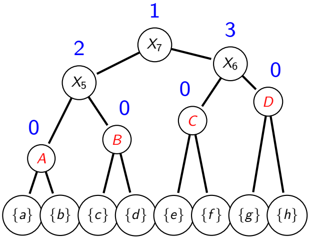

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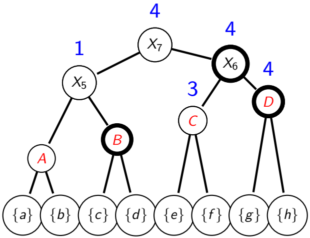

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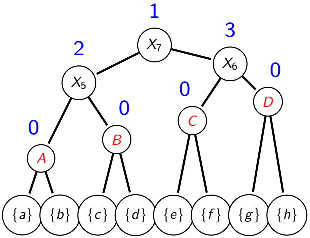

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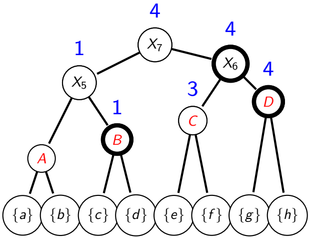

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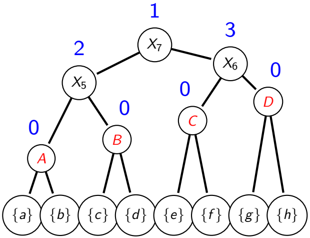

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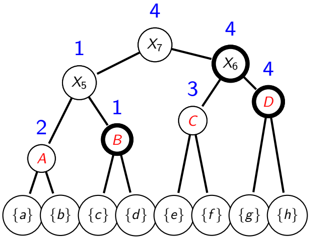

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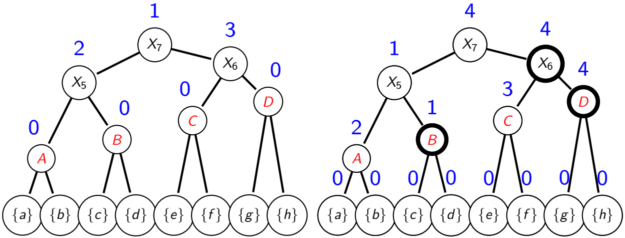

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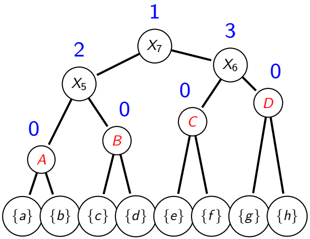

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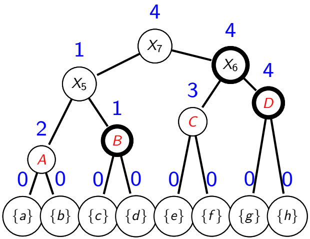

ξ_s

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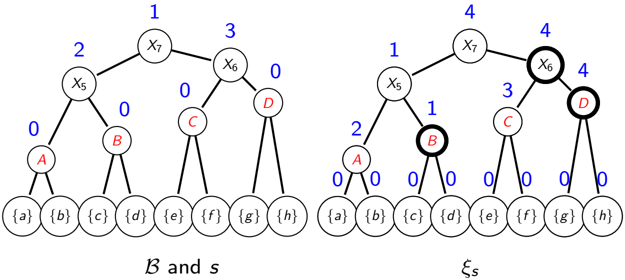

$\mathcal{B}$ and s

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Estimated sequence of minima for s: $\mathcal{S}_f = (B, A, C, D)$

Watersheding operator

Definition (watersheding)

Let f be a map from the set of edges E into $\mathbb{R}$ and let $\mathcal{S}$ be the estimated sequence of minima of f and let ϵ be the extinction map for $\mathcal{S}$. The *watersheding of f* is the map $\omega(f)$ from E into $\mathbb{R}$ such that, for any edge u:

- $\omega(f)(u) = min\{\epsilon(R) \mid R \text{ } is \text{ } a \text{ } child \text{ } of \text{ } R_u\}$,

where R_u is the region of $\mathcal{B}$ whose building edge is u.

Watersheding operator: properties

Property (Idempotence)

Let f be a map from E into $\mathbb{R}$. The watersheding $\omega(\omega(f))$ of $\omega(f)$ is equal to the watersheding $\omega(f)$ of f.

Property (Invariance domain)

Let $\mathcal{H}$ be a hierarchy and let f be the saliency map of $\mathcal{H}$. The watersheding of f is equal to f if and only if $\mathcal{H}$ is a hierarchical watershed of (G, w).

Property (Solution to the problem of recognizing hierarchical watersheds (M19))

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[M19] D. Maia, J.Cousty, L. Najman, B. Perret. Recognizing hierarchical watersheds. 2019.

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Outlines

1 Hierarchical watersheds

2 Watersheding transform

3 Conclusion and perspectives

Conclusion and perspectives

Summary

- *Introduction of an idempotent operator which converts any hierarchy into a hierarchical watersheds*

Perspectives

- *Extension to arbitrary weighted graphs*
- *Open question: does the watersheding optimize any objective function?*

Watersheding operator: applications

2. Regularization of hierarchies based on non-increasing attributes:
    - *e.g.*, circularity

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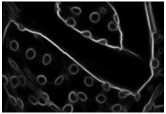

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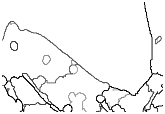

$\mathcal{H}_c$

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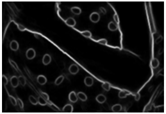

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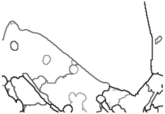

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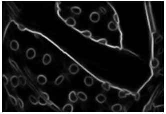

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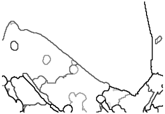

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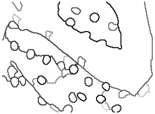

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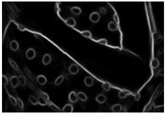

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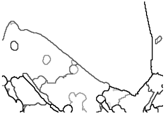

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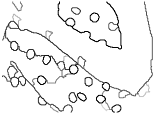

$\mathcal{H}_w$

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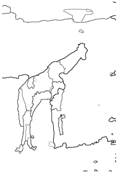

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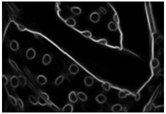

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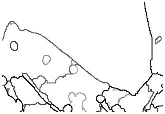

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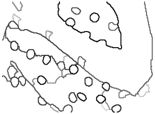

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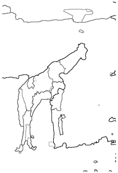

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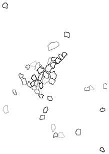

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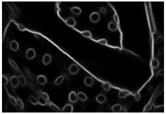

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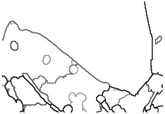

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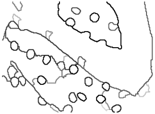

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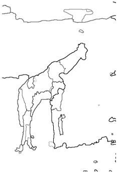

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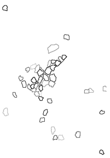

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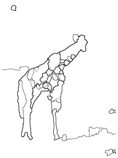

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