On the probabilities of hierarchical watersheds

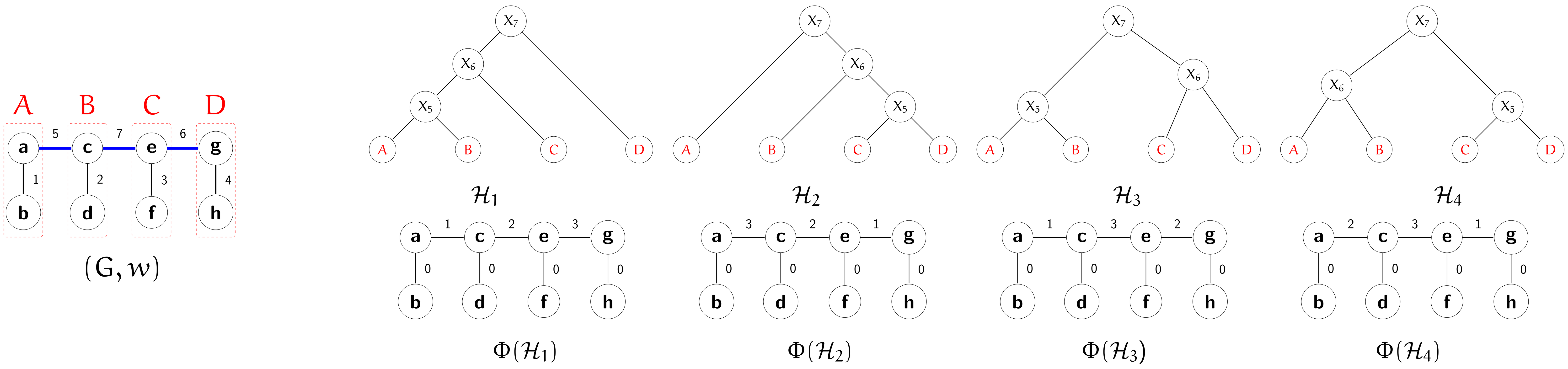
Deise Santana Maia, Jean Cousty, Laurent Najman and Benjamin Perret

UNIVERSITÉ PARIS-EST
LABORATOIRE D'INFORMATIQUE GASPARD-MONGE

Introduction

- **Hierarchical watersheds** [1,2] are obtained by iteratively merging the regions of a watershed segmentation. This merging procedure is guided by a total ordering on the regional minima of an image (gradient), which can be represented as a graph. The **probability of a hierarchical watershed** is associated to the number of orderings of minima of a graph that could be used to obtain this hierarchy.
- **Contributions:** the introduction of the notion of probability of a hierarchical watershed, an efficient algorithm (see article) to obtain the probability of any hierarchical watershed, a characterization and an algorithm to obtain the most probable hierarchical watersheds of a weighted graph.

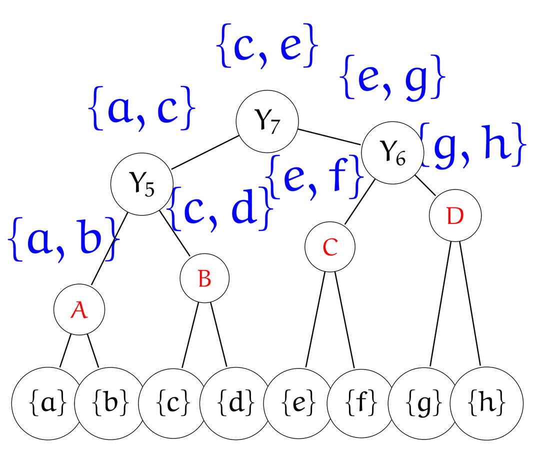
Hierarchical watersheds and Saliency maps



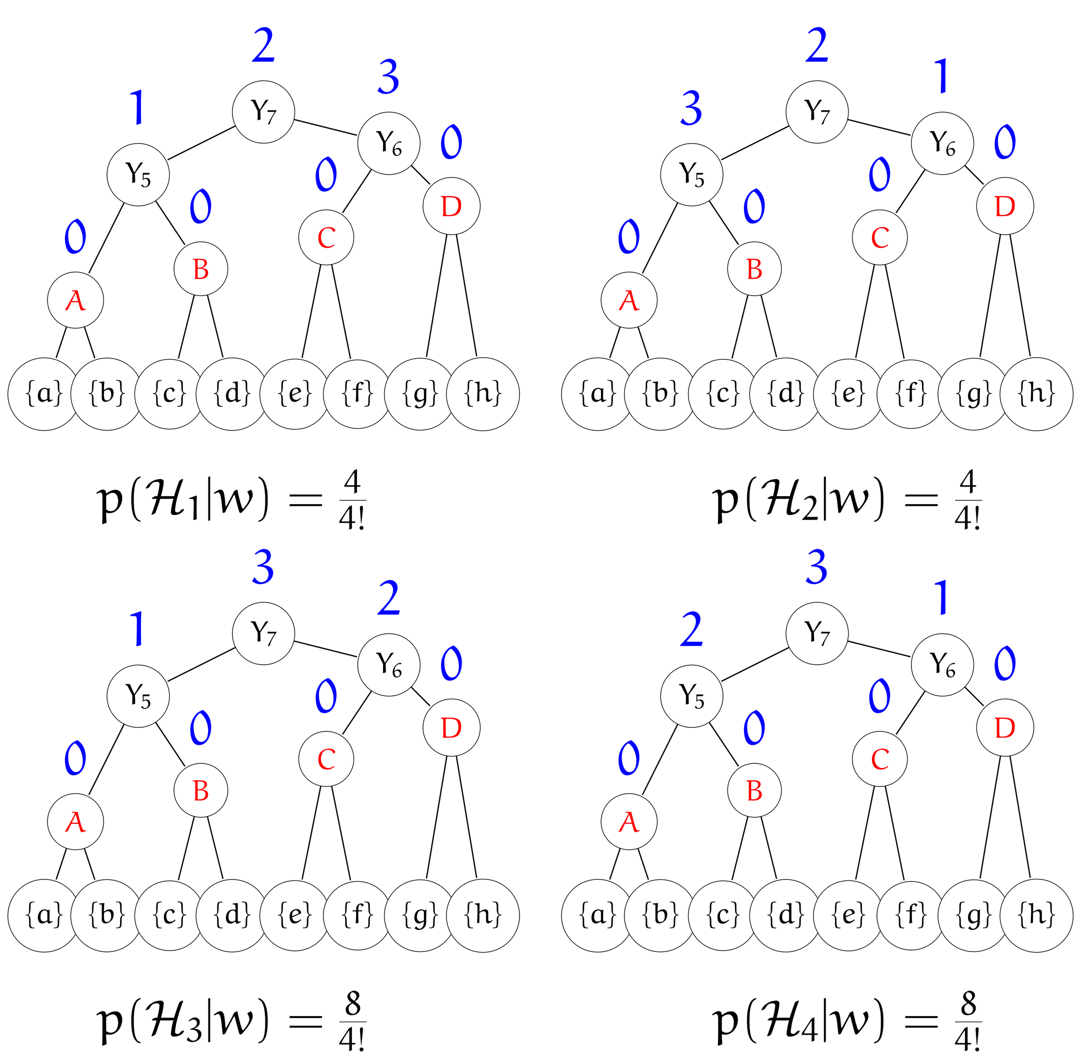
A weighted graph (G, w) with four minima (in red) and three watershed-cut edges (in blue), the four hierarchical watersheds of (G, w) and their saliency maps. Each hierarchical watershed can be obtained from more than one sequence of minima of w. For example, $\mathcal{H}_1$ can be obtained from the following sequences: (A, B, C, D), (A, B, D, C), (B, A, C, D) and (B, A, D, C)

Binary partition hierarchy and Probability of hierarchical watersheds

- (G, w): tree with pairwise distinct edge weights containing n minima
- $\mathcal{H}$ and $\Phi(\mathcal{H})$: hierarchical watershed of (G, w) and saliency map of $\mathcal{H}$
- $S_w(\mathcal{H})$: set of sequences of minima of w such that, for any sequence $\mathcal{S}$ in $S_w(\mathcal{H})$, $\mathcal{H}$ is the hierarchical watershed of (G, w) for $\mathcal{S}$
- $\mathcal{M}_w$: set of all possible sequences of minima of w
- The **probability of $\mathcal{H}$ knowing** w, denoted by $p(\mathcal{H}|w)$, is the ratio $\frac{|S_w(\mathcal{H})|}{|\mathcal{M}_w|}$, and can be obtained through the **binary partition hierarchy (by altitude ordering)** $\mathcal{B}$ **of** (G, w)
- $\mathcal{B}$: constructed by merging the singletons of G considering an increasing order of weights in w. For each edge $u = \{x, y\}$, we denote by R_u the region obtained after merging the regions that contain x and y
- m: the number of edges u in E such that $\Phi(\mathcal{H})(u) > \max\{\Phi(\mathcal{H})(v), v \in E \mid R_v \subset R_u\}$.
- **Theorem 1:** The probability of $\mathcal{H}$ knowing w is $p(\mathcal{H} \mid w) = \frac{2^m}{|\mathcal{M}_w|}$

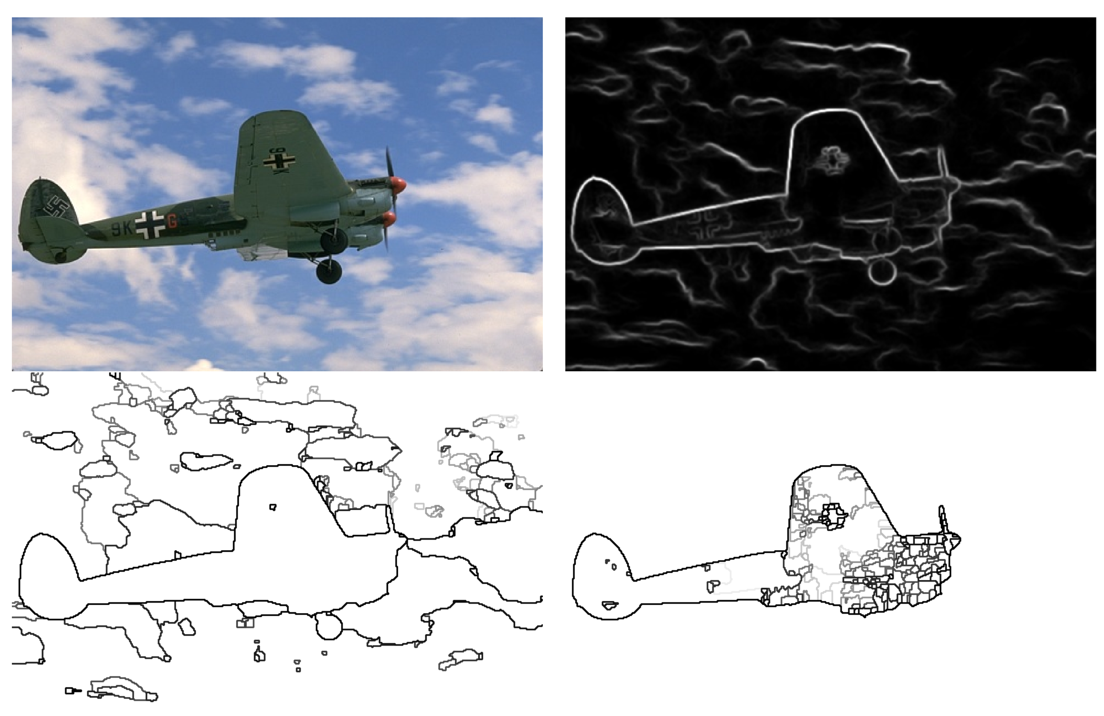


The binary partition hierarchy $\mathcal{B}$ of (G, w)



Most probable hierarchical watersheds

- ℓ: number of watershed-cut edges of w
- k: number of watershed-cut edges u such that there is no watershed-cut edge v such that $R_v \subset R_u$
- **Corollary 2:** The tight upper and lower upper bounds on the probability of $\mathcal{H}$ knowing w are respectively $\frac{2^\ell}{|\mathcal{M}_w|}$ and $\frac{2^k}{|\mathcal{M}_w|}$
- $\mathcal{H}$ is a **most probable hierarchical watershed of** (G, w) if, for any hierarchical watershed $\mathcal{H}'$ for (G, w), $p(\mathcal{H} \mid w) \geq p(\mathcal{H}' \mid w)$
- **Theorem 3:** The three following statements are equivalent: (1) $\mathcal{H}$ is a most probable hierarchical watershed of (G, w); (2) the edge weights in $\Phi(\mathcal{H})$ are increasing on $\mathcal{B}$; and (3) each non-leaf region of $\mathcal{H}$ is a region of $\mathcal{B}$



An image I, a gradient Grad of I and the saliency maps of two of the most probable hierarchical watersheds of Grad.

[1] J. Cousty, L. Najman, and B. Perret. Constructive links between some morphological hierarchies on edge-weighted graphs. In ISMM, pages 86–97. Springer, 2013.

[2] L. Najman, J. Cousty, and B. Perret. Playing with Kruskal: algorithms for morphological trees in edge-weighted graphs. In ISMM, pages 135–146. Springer, 2013.