Internship

Title : The Container Method in Combinatorics and Geometry.

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Context : The theory of ϵ-nets and ϵ-approximations is fundamental in understanding the use of random sampling in probability, combinatorics and algorithms. Given a finite set system $(X, \mathcal{R})$ and a parameter $\epsilon > 0$, an ϵ-net is a set $N \subseteq X$ such that $N \cap R \neq \emptyset$ for all sets $R \in \mathcal{R}$ with $|R| \geq \epsilon \cdot |X|$. Similarly, a set $A \subseteq X$ is an ϵ-approximation if for each $R \in \mathcal{R}$, we have

$$\left| \frac{|R|}{|X|} - \frac{|R \cap A|}{|A|} \right| \leq \epsilon.$$

The key question in the study of ϵ-nets and ϵ-approximations is bounding their sizes for basic combinatorial and geometric set systems. See the recent survey [3] on this topic.

Main Problem : Let P be a set of n points in the plane, and $\mathcal{R}$ be the *primal set system* on P induced by containment by lines in the plane. In other words, a set $R \subseteq P$ is in $\mathcal{R}$ if and only if there exists a line l in the plane containing precisely the points of R on it, i.e., $R = l \cap P$. Then the open problem is to prove the following conjecture :

> **Given any parameter $\epsilon > 0$, there exists a set P of points in the plane such that any ϵ-net for the primal set system induced on P by lines in the plane must have size $\Omega\left(\frac{1}{\epsilon}\log\frac{1}{\epsilon}\right)$.**

Previous approach : This problem is closely related to the density version of the Hales-Jewett theorem, for which a combinatorial proof was given by Gowers, Tao and others in the polymath project [4]. The Hales-Jewett theorem was then used by N. Alon [1] to prove a lower-bound of $\Omega\big(\frac{1}{\epsilon}w(\frac{1}{\epsilon})\big)$ for ϵ-nets for lines, where $w(s)$ is the minimum number k so that $k^{A_k(2)} > s$, where A_k is the k-th function in the Ackermann hierarchy. This is *very slightly* super-linear.

New approach : Very recently, Balog and Solymosi [2] used a powerful new technique in combinatorics, called the *container method*, to give a substantial improvement to Alon's lower-bound, improving it to $\Omega\left(\frac{1}{\epsilon}\left(\log\frac{1}{\epsilon}\right)^{1/5}\right)$. However, there are many parameters in the proof of the 'container method', and it is possible that the use of this method can be refined to prove an optimal lower-bound of $\Omega(\frac{1}{\epsilon}\log\frac{1}{\epsilon})$. That is the aim of this project.

Plan : The project will involve two stages. The first stage is to understand the Hales-Jewett theorem and its use in the lower-bound of Alon, understand the proof and the ideas behind the container method, and then reading and learning the use of this method by Balogh and Solymosi to improve the lower-bound of Alon. The second stage will then be to work on improving this lower-bound.

References

[1] N. Alon. A non-linear lower bound for planar epsilon-nets. *Discrete Comput. Geom.*, 47 :235–244, 2012.

[2] J. Balogh and J. Solymosi. On the number of points in general position in the plane. *ArXiv e-prints*, 1704.0509.

[3] N. H. Mustafa and K. Varadarajan. Epsilon-approximations and epsilon-nets. In J. E. Goodman, J. O'Rourke, and C. D. Tóth, editors, *Handbook of Discrete and Computational Geometry*, CRC Press LLC, 2017.

[4] D. H. J. Polymath. A new proof of the density Hales-Jewett theorem. *Annals of Mathematics*, 175(3) : 1283–1327.